

THE N -POINT CORRELATION OF QUADRATIC FORMS.

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ABSTRACT. In this paper we investigate the distribution of the set of values of a quadratic form Q , at integral points. In particular we are interested in the n -point correlations of the this set. The asymptotic behaviour of the counting function that counts the number of n -tuples of integral points (v_1, \dots, v_n) , with bounded norm, such that the $n-1$ differences $Q(v_1) - Q(v_2), \dots, Q(v_{n-1}) - Q(v_n)$, lie in prescribed intervals is obtained. The results are valid provided that the quadratic form has rank at least 5, is not a multiple of a rational form and n is at most the rank of the quadratic form. For certain quadratic forms satisfying Diophantine conditions we obtain a rate for the limit. The proofs are based on those in the recent preprint ([GM13]) of F. Götze and G. Margulis, in which they prove an ‘effective’ version of the Oppenheim Conjecture. In particular, the proofs rely on Fourier analysis and estimates for certain theta series.

1. INTRODUCTION.

1.1. Background. Let $Q : \mathbb{R}^d \rightarrow \mathbb{R}$ be a quadratic form. It is interesting to understand the distribution of the set $Q(\mathbb{Z}^d)$ inside \mathbb{R} . If there exists a multiple of Q such that its coefficients are all rational, then Q is called a rational form. In the case when Q is rational, $Q(\mathbb{Z}^d)$ is a discrete set inside \mathbb{R} . When Q is not a rational form, Q is called an irrational form. For irrational forms, the first milestone in understanding the distribution of the set $Q(\mathbb{Z}^d)$ inside \mathbb{R} was reached by G. Margulis in [Mar89] when he provided a proof (shortly afterwards, refined by S.G. Dani and G. Margulis in [DM89]) of the ‘Oppenheim Conjecture’. The modern statement of which is as follows: if $d \geq 3$ and Q is a nondegenerate, irrational and indefinite form, then $Q(\mathbb{Z}^d)$ is dense in \mathbb{R} .

Once it is known that $Q(\mathbb{Z}^d)$ is dense in \mathbb{R} , one can ask for a more precise answer to the question of how $Q(\mathbb{Z}^d)$ is distributed in \mathbb{R} . Let $I \subset \mathbb{R}$ be any interval and $E_Q(I, T) = \{v \in \mathbb{R}^d : Q(v) \in I, \|v\| \leq T\}$, then one can ask for an asymptotic formula for the size of the set $\mathbb{Z}^d \cap E_Q(I, T)$. The first results in this direction were obtained by Dani-Margulis in [DM93] who proved, if $d \geq 3$ and Q is a nondegenerate, irrational and indefinite form and I is any interval, then

$$\lim_{T \rightarrow \infty} \frac{|\mathbb{Z}^d \cap E_Q(I, T)|}{\text{Vol}(E_Q(I, T))} \geq 1.$$

The situation regarding the upper bounds is more delicate and this was dealt with by the work of A. Eskin, G. Margulis and S. Mozes in [EMM98] who proved that if $d \geq 5$ and Q is a nondegenerate, irrational and indefinite form and I is any interval, then

$$(1.1) \quad \lim_{T \rightarrow \infty} \frac{|\mathbb{Z}^d \cap E_Q(I, T)|}{\text{Vol}(E_Q(I, T))} = 1.$$

It should be noted that the actual results from [DM93] and [EMM98] are more general than stated above. The situation regarding the upper bounds for the case when $d = 3$ or 4 is particularly interesting and is also considered in [EMM98]. In the cases when Q has signature $(2, 1)$ or $(2, 2)$ no asymptotic formula of the form (1.1) is possible for general quadratic forms, since in these cases there exist examples of quadratic forms for which (1.1) fails. In [EMM05], quadratic forms of signature $(2, 2)$ satisfying a slightly modified version of (1.1) are characterised by certain Diophantine conditions. The work of Eskin-Margulis-Mozes can be interpreted as providing conditions which ensure the set $Q(\mathbb{Z}^d)$ is equidistributed in \mathbb{R} .

One can ask still finer questions about the distribution of the set $Q(\mathbb{Z}^d)$. Let e_1, \dots, e_{nd} be the standard basis of \mathbb{R}^{nd} , let p_1, \dots, p_n denote the projections onto $\langle e_1, \dots, e_d \rangle, \dots, \langle e_{(n-1)d+1}, \dots, e_{nd} \rangle$ respectively. For $v \in \mathbb{R}^{nd}$ and $1 \leq i \leq n$, we will write $v_i = p_i(v)$. Let I_1, \dots, I_{n-1} be intervals and

$$P_Q^n(I_1, \dots, I_{n-1}, T) = \{v \in \mathbb{R}^{nd} : Q(v_1) - Q(v_2) \in I_1, \dots, Q(v_{n-1}) - Q(v_n) \in I_{n-1}, \|v\| \leq T\}.$$

In order to understand the n -point correlations of the set $Q(\mathbb{Z}^d)$, one asks for an asymptotic formula for the size of the set $\mathbb{Z}^{nd} \cap P_Q^n(I_1, \dots, I_{n-1}, T)$. A more general problem about the distribution of values at integral points of systems of quadratic forms was studied by W. Müller in [Mül08]. In particular, it follows from Theorem 1 of

[Mül08] that if $n \geq 2$, $4n \leq d$ and Q is a nondegenerate and irrational form, then

$$(1.2) \quad \lim_{T \rightarrow \infty} \frac{|\mathbb{Z}^{nd} \cap P_Q^n(I_1, \dots, I_{n-1}, T)|}{\text{Vol}(P_Q^n(I_1, \dots, I_{n-1}, T))} = 1.$$

When $n = 2$ and $d \geq 3$, it is easy to see that (1.2) follows from the main Theorem of [EMM98]. For positive definite forms the n -point correlation problem was also studied by Müller. In [Mül11], Müller obtains the following result: if $d \geq 4$ and Q is a nondegenerate, irrational and positive definite form, then (1.2) holds for every n . In [Mül11] Müller formulates the problem in slightly different language, but it is easily seen to be equivalent to the form stated here up to a change of variables and modifications of the norms involved. The main result of this paper extends the results of Müller to a larger range of n for indefinite forms.

1.2. Statement of results. Using the notation from the previous subsection we can now state the main results.

Theorem 1.1. *Suppose that Q is not a multiple of a rational form and $d \geq 5$ and $2 < n \leq d$. Then, for any intervals I_1, \dots, I_{n-1} ,*

$$\lim_{T \rightarrow \infty} \frac{|\mathbb{Z}^{nd} \cap P_Q^n(I_1, \dots, I_{n-1}, T)|}{\text{Vol}(P_Q^n(I_1, \dots, I_{n-1}, T))} = 1.$$

Moreover, there exists a positive constant $C_{Q,n}$, depending only on Q and n , such that for any intervals I_1, \dots, I_{n-1} ,

$$\lim_{T \rightarrow \infty} \frac{|\mathbb{Z}^{nd} \cap P_Q^n(I_1, \dots, I_{n-1}, T)|}{T^{nd-2(n-1)}} = C_{Q,n} \prod_{i=1}^{n-1} |I_i|.$$

For quadratic forms Q satisfying the following Diophantine condition it is possible to prove an effective version of Theorem 1.1. Let Q also denote the symmetric $d \times d$ matrix that is associated to the quadratic form Q . Let $0 < \kappa < 1$ and $A > 0$, say that Q is of type (κ, A) if for every $M \in \text{Mat}_d(\mathbb{Z})$ and $q \in \mathbb{Z} \setminus \{0\}$ we have

$$\inf_{t \in [1,2]} \|Mq^{-1} - tQ\| \geq Aq^{-1-\kappa}.$$

The size of κ depends on how well Q can be approximated by a rational matrix, if κ is close to 1, then Q is in some sense close to a rational matrix.

Theorem 1.2. *Suppose that Q is of Diophantine type (κ, A) and $d \geq 5$ and $2 < n \leq d$. Let $\delta(\kappa) = \frac{2(d-4)(1-\kappa)}{(1+nd)(d+1+\kappa)}$. Then, for any intervals I_1, \dots, I_{n-1} there exists $T_0 > 0$ and a constant C such that for all $T \geq T_0$,*

$$\left| \frac{|\mathbb{Z}^{nd} \cap P_Q^n(I_1, \dots, I_{n-1}, T)|}{\text{Vol}(P_Q^n(I_1, \dots, I_{n-1}, T))} - 1 \right| \leq C \log^{n-1}(T) T^{-\delta(\kappa)}.$$

Remark 1.3. The constant C appearing in Theorem 1.2 depends on Q, n, A and the intervals I_1, \dots, I_{n-1} .

Remark 1.4. In Theorem 1.2 we use $\|\cdot\|$ to denote the Euclidean norm. The exponent $\delta(\kappa)$ depends on the choice of norm and is possibly non optimal. If the maximum norm was chosen, the bounds in subsection 5.2 could be improved, and $\delta(\kappa)$ could be replaced with $\delta'(\kappa) = \frac{2(1-\kappa)}{(d+1+\kappa)}$ at the cost of a factor of $\log^{nd}(T)$ appearing.

Remark 1.5. The second parts of Theorems 1.1 and 1.2 follow easily from the first parts and the following assertion: For any intervals I_1, \dots, I_{n-1} there exists a positive constant $C_{Q,n}$, depending only on Q and n , such that

$$\lim_{T \rightarrow \infty} \frac{1}{T^{nd-2(n-1)}} \text{Vol}(P_Q^n(I_1, \dots, I_{n-1}, T)) = C_{Q,n} \prod_{i=1}^{n-1} |I_i|.$$

This statement is proved in Corollary 5.5.

1.3. The Berry-Tabor Conjecture. For positive definite forms there is a similar problem about the n -point correlations of the normalised values of Q at integral points. This problem is discussed in [Mül08] and is interesting because it is related to the so called Berry-Tabor Conjecture (see [BT77]). A special case of this Conjecture states that the spacings of eigenvalues of the Laplacian on ‘generic’ multidimensional tori should have a Poisson distribution. This problem has been studied in [Sar97] by P. Sarnak, in [Van99] and [Van00] by J. VanderKam and in [Mar02] by J. Marklof.

1.4. Outline of paper and summary of the methods. One can try to prove Theorems 1.1 and 1.2 by using the theory of unipotent flows, in analogy to what was done in [EMM98]. The problem one encounters, is that the subgroup of linear transformations of \mathbb{R}^{nd} stabilising the quadratic forms $Q(v_1) - Q(v_2), \dots, Q(v_{n-1}) - Q(v_n)$ is $SO(Q)^n$ and this seems too small to obtain the required statements. If one had access to a precise quantitative equidistribution statement, in the form of an explicit rate for the limit in (1.1), one could hope to prove results like Theorems 1.1 and 1.2. Unfortunately, since the results of [EMM98] relied on the equidistribution of unipotent flows, no good error term was available. However, recently, F. Götze and G. Margulis proved such a statement in the preprint [GM13] (see also [GM10] for an older version). Their methods do not rely on the equidistribution of unipotent flows. Instead they use Fourier analysis to reduce the problem to one of obtaining asymptotic estimates for certain theta series. In order to estimate these theta series, they use some of the techniques developed in [EMM98], in particular the crux of their proof relies on a non divergence statement about average of the translates of orbits of certain compact subgroups in the space of lattices. One cannot apply the results of [GM13] directly, since in order to do this one would need the error to be uniform across all intervals. However, the proofs of Theorems 1.1 and 1.2 are based on the methods of [GM13].

The object of interest is

$$R\left(\mathbb{1}_{P_Q^n(I_1, \dots, I_{n-1}, T)}\right) = \sum_{v \in \mathbb{Z}^{nd}} \mathbb{1}_{P_Q^n(I_1, \dots, I_{n-1}, T)}(v) - \int_{\mathbb{R}^{nd}} \mathbb{1}_{P_Q^n(I_1, \dots, I_{n-1}, T)}(v) dv,$$

where, here and throughout the rest of the paper, for any set S , $\mathbb{1}_S$ stands for the characteristic function of the set S . Theorems 1.1 and 1.2 follow from suitable bounds for $|R(\mathbb{1}_{P_Q^n(I_1, \dots, I_{n-1}, T)})|$. To obtain these bounds, the function $\mathbb{1}_{P_Q^n(I_1, \dots, I_{n-1}, T)}$ is replaced with a smoothened version at the cost of ‘smoothing errors’ which can be estimated in terms of volumes of certain regions of \mathbb{R}^{nd} . This is carried out in subsections 2.1 and 5.1. The next step is to use Fourier analysis to transfer the problem into the ‘frequency domain’. After taking Fourier transforms, the smoothened version of $|R(\mathbb{1}_{P_Q^n(I_1, \dots, I_{n-1}, T)})|$ can be estimated by considering an integral over the ‘frequency domain’, $\omega \in \mathbb{R}^{n-1}$, of the difference between a theta series, $\theta(\omega)$ and its corresponding smooth version, $\vartheta(\omega)$ (see (2.9)). This step is carried out in subsection 2.2. In order to estimate the integral, the domain of integration is split into two parts, namely a neighbourhood of the origin and its complement.

The integral over the region bounded away from the origin is dealt with by considering the integral of $\theta(\omega)$ and the integral of $\vartheta(\omega)$ separately. The integral of $\theta(\omega)$ contributes the main term in the bound for $|R(\mathbb{1}_{P_Q^n(I_1, \dots, I_{n-1}, T)})|$ and it contains the arithmetic information about Q . This term is dealt with in subsection 3.3. The integral of $\vartheta(\omega)$ only contributes a lower order term to the bound for $|R(\mathbb{1}_{P_Q^n(I_1, \dots, I_{n-1}, T)})|$ and is dealt with in subsection 3.1. These two integrals can be estimated using techniques and results from [GM13]. The reason for this, is that $\theta(\omega)$ and $\vartheta(\omega)$ can be written as a product of n sums/integrals of the form studied in [GM13] (see (2.10) and (2.11)).

The integral, over the neighbourhood of the origin, is dealt with in subsection 3.2. This term contributes a lower order term, but it grows with n , faster than the main term. For $n > d$ this term dominates the main term, explaining why the assumption $n \leq d$ is needed. The reason for this, is that here we consider the difference, $\theta(\omega) - \vartheta(\omega)$. Poisson summation is used to convert this into a sum over $\mathbb{Z}^{nd} \setminus \{0\}$, the problem that arises is that for $m \in \mathbb{Z}^{nd} \setminus \{0\}$ we can still have $m_i = 0$ for some $1 \leq i \leq d$. Therefore, although it is still possible to take advantage of the fact that the sum obtained by Poisson summation can be written as a product of n sums, an additional argument is needed to deal with the fact that 0 could be included in each of the sums in the product.

Finally in Section 4 all of the bounds are collected and Theorems 1.1 and 1.2 are proved. The bounds obtained in Section 3 depend on the L^1 norm of a certain function which depends on a smoothing parameter. In order to prove Theorem 1.2 we need a precise estimates for this norm in terms of the smoothing parameter. This is carried out in subsection 5.2.

2. SET UP.

For the rest of the paper let n and d be natural numbers with $n \leq d$. In the case when $n = 2$, there is only one quadratic form and the conclusions of Theorems 1.1 and 1.2 follow from the results of [EMM98] and [GM13]. Hence, throughout the rest of the paper we suppose that $n \geq 3$. For $1 \leq i \leq n-1$, fix intervals I_i and $Q : \mathbb{R}^d \rightarrow \mathbb{R}$ a nondegenerate quadratic form, suppose that $d \geq 5$ and keep the notation from the introduction. Let $Q_+ = Q^2$, hence Q_+ corresponds to a positive definite quadratic form. Let $\text{sp}(Q)$ denote the spectrum of Q , $\lambda_{\min} = \min_{\lambda \in \text{sp}(Q)} |\lambda|$ and $\lambda_{\max} = \max_{\lambda \in \text{sp}(Q)} |\lambda|$. Since the problem is unaffected by rescaling Q , we may suppose that $\lambda_{\min} \geq n-1$, this supposition will be used in the proof of Lemma 3.13. Define $B(T) = \{v \in \mathbb{R}^{nd} : \|v\| \leq T\}$ and $B_\infty(T) = \{v \in \mathbb{R}^{nd} : \|v\|_\infty \leq T\}$, where we use $\|\cdot\|$ to denote the Euclidean norm and $\|\cdot\|_\infty$ to denote the

maximum norm. Let

$$P_Q^n(I_1, \dots, I_{n-1}) = \{v \in \mathbb{R}^{nd} : Q(v_1) - Q(v_2) \in I_1, \dots, Q(v_{n-1}) - Q(v_n) \in I_{n-1}\}.$$

Note that $P_Q^n(I_1, \dots, I_{n-1}, T) = P_Q^n(I_1, \dots, I_{n-1}) \cap B(T)$. As is standard, we use the notation \hat{f} to denote the Fourier transform of a function f . We will also make heavy use the Vinogradov asymptotic notation $f(s) \ll g(s)$, which means that there exists some constant $C > 0$ such that $f(s) \leq Cg(s)$ for all values of s indicated. The constant C will be independent of those parameters but will usually depend on d, n, Q and the intervals I_1, \dots, I_{n-1} .

2.1. Smoothing. For any $i \in \mathbb{N}$, let $k^i = k^i(v) dv$ be a probability measure on \mathbb{R}^i with the properties that it is symmetric around 0, $k^i(\{v \in \mathbb{R}^i : \|v\| \leq 1\}) = 1$ and

$$(2.1) \quad |\hat{k}^i(v)| \leq \exp(-c\sqrt{\|v\|})$$

for some positive constant c and all $v \in \mathbb{R}^i$. For any $\tau > 0$, let k_τ^i denote the rescaled measure such that $k_\tau^i(A) = k^i(\tau^{-1}A)$ for any measurable set A . Note that (2.1) implies that

$$(2.2) \quad |\hat{k}_\tau^i(v)| \leq \exp(-c\sqrt{\tau\|v\|}).$$

For an interval $I = [a, b]$ and $\epsilon \in \mathbb{R}$, define $I^\epsilon = [a - \epsilon, b + \epsilon]$. For any $\tau > 0$, $T > 0$ and $v \in \mathbb{R}^{nd}$, let

$$w_{\pm\tau}(v) = \mathbb{1}_{B(1\pm\tau)} * k_\tau^{nd}(v) \quad \text{and} \quad w_{\pm\tau, T}(v) = w_{\pm\tau}(T^{-1}v).$$

For any $\epsilon > 0$ and $\omega \in \mathbb{R}^{n-1}$, let

$$S_{\pm\epsilon}(\omega) = \mathbb{1}_{I_1^{\pm\epsilon} \times \dots \times I_{n-1}^{\pm\epsilon}} * k_\epsilon^\Pi(\omega),$$

where $k_\epsilon^\Pi(\omega) = k_\epsilon^1(\langle \omega, e_1 \rangle) \dots k_\epsilon^1(\langle \omega, e_{n-1} \rangle)$. For $v \in \mathbb{R}^{nd}$, let

$$S_{\pm\epsilon}^Q(v) = S_{\pm\epsilon}(Q(v_1) - Q(v_2), \dots, Q(v_{n-1}) - Q(v_n)).$$

For a measurable function f on \mathbb{R}^{nd} define

$$(2.3) \quad R(f) = \sum_{v \in \mathbb{Z}^{nd}} f(v) - \int_{\mathbb{R}^{nd}} f(v) dv.$$

Note that $R(f)$ is only well defined if both the quantities on the right hand side of (2.3) are finite. Let ν_T and $\nu_{\tau, T}$ be measures on \mathbb{R}^{nd} and \mathbb{R}^{n-1} respectively, defined by

$$\int_{\mathbb{R}^{nd}} f d\nu_T = \int_{\mathbb{R}^{nd}} f(T^{-1}v) \mathbb{1}_{P_Q^n(I_1, \dots, I_{n-1})}(v) dv$$

and

$$\int_{\mathbb{R}^{n-1}} f d\nu_{\tau, T} = \int_{\mathbb{R}^{nd}} f(Q(v_1) - Q(v_2), \dots, Q(v_{n-1}) - Q(v_n)) w_{\pm\tau, T}(v) dv.$$

In the next two Lemmas we approximate $R(\mathbb{1}_{P_Q^n(I_1, \dots, I_{n-1})} \mathbb{1}_{B(T)})$ by a smoothened version.

Lemma 2.1. *For all $\tau > 0$ and $T > 0$,*

$$\left| R(\mathbb{1}_{P_Q^n(I_1, \dots, I_{n-1})} \mathbb{1}_{B(T)}) \right| \leq \max_{\pm\tau} \left| R(\mathbb{1}_{P_Q^n(I_1, \dots, I_{n-1})} w_{\pm\tau, T}) \right| + \int_{\mathbb{R}^{nd}} (\mathbb{1}_{B(1+2\tau)} - \mathbb{1}_{B(1-2\tau)}) d\nu_T.$$

Proof. Define a measure μ_T on \mathbb{R}^{nd} , by

$$\int_{\mathbb{R}^{nd}} f d\mu_T = \sum_{v \in \mathbb{Z}^{nd}} f(T^{-1}v) \mathbb{1}_{P_Q^n(I_1, \dots, I_{n-1})}(v).$$

Define functions on \mathbb{R}^{nd} by $f = \mathbb{1}_{B(1)}$ and $f_{\pm\tau} = \mathbb{1}_{B(1\pm\tau)}$. Note that

$$(2.4) \quad \left| R(\mathbb{1}_{P_Q^n(I_1, \dots, I_{n-1})} \mathbb{1}_{B(T)}) \right| = \left| \int_{\mathbb{R}^{nd}} f d(\mu_T - \nu_T) \right|.$$

From the definition of k_τ^{nd} it follows that

$$f_{-2\tau} \leq f_{-\tau} * k_\tau^{nd} \leq f \leq f_{+\tau} * k_\tau^{nd} \leq f_{+2\tau}.$$

Since all the functions in the previous inequality are bounded and have compact support and the measure $\mu_T - \nu_T$ is locally finite, by integrating with respect to $\mu_T - \nu_T$ we obtain

$$\begin{aligned} \int_{\mathbb{R}^{nd}} f d(\mu_T - \nu_T) &\leq \int_{\mathbb{R}^{nd}} f_{+\tau} * k_\tau^{nd} d(\mu_T - \nu_T) + \int_{\mathbb{R}^{nd}} (f_{+\tau} * k_\tau^{nd} - f) d\nu_T \\ &\leq \int_{\mathbb{R}^{nd}} f_{+\tau} * k_\tau^{nd} d(\mu_T - \nu_T) + \int_{\mathbb{R}^{nd}} (f_{+2\tau} - f_{-2\tau}) d\nu_T. \end{aligned}$$

Similarly

$$\int_{\mathbb{R}^{nd}} f d(\mu_T - \nu_T) \geq \int_{\mathbb{R}^{nd}} f_{-\tau} * k_\tau^{nd} d(\mu - \nu) - \int_{\mathbb{R}^{nd}} (f_{+2\tau} - f_{-2\tau}) d\nu_T.$$

In view of (2.4) and the definition of $w_{\pm\tau, T}$ the conclusion of the Lemma follows from the previous two inequalities. \square

Lemma 2.2. *For all $\epsilon > 0$, $\tau > 0$ and $T > 0$,*

$$\left| R \left(\mathbb{1}_{P_Q^n(I_1, \dots, I_{n-1})} w_{\pm\tau, T} \right) \right| \leq \max_{\pm\epsilon} \left| R \left(S_{\pm\epsilon}^Q w_{\pm\tau, T} \right) \right| + \int_{\mathbb{R}^{n-1}} \left(\mathbb{1}_{I_1^{2\epsilon} \times \dots \times I_{n-1}^{2\epsilon}} - \mathbb{1}_{I_1^{-2\epsilon} \times \dots \times I_{n-1}^{-2\epsilon}} \right) d\nu_{\tau, T}.$$

Proof. Define a measure $\mu_{\tau, T}$ on \mathbb{R}^{n-1} , by

$$\int_{\mathbb{R}^{n-1}} f d\mu_{\tau, T} = \sum_{v \in \mathbb{Z}^{nd}} f(Q(v_1) - Q(v_2), \dots, Q(v_{n-1}) - Q(v_n)) w_{\pm\tau, T}(v).$$

Define functions on \mathbb{R}^{nd} by $f = \mathbb{1}_{I_1 \times \dots \times I_{n-1}}$ and $f_{\pm\epsilon} = \mathbb{1}_{I_1^{\pm\epsilon} \times \dots \times I_{n-1}^{\pm\epsilon}}$. Note that

$$(2.5) \quad \left| R \left(\mathbb{1}_{P_Q^n(I_1, \dots, I_{n-1})} w_{\pm\tau, T} \right) \right| = \left| \int_{\mathbb{R}^{n-1}} f d(\mu_{\tau, T} - \nu_{\tau, T}) \right|.$$

From the definition of k_ϵ^Π it follows that

$$f_{-2\epsilon}(\omega) \leq f_{-\epsilon} * k_\epsilon^\Pi(\omega) \leq f(\omega) \leq f_{+\epsilon} * k_\epsilon^\Pi(\omega) \leq f_{+2\epsilon}.$$

Since all the functions in the previous inequality are bounded and have compact support and the measure $\mu_{\tau, T} - \nu_{\tau, T}$ is locally finite, by integrating with respect to $\mu_{\tau, T} - \nu_{\tau, T}$ as in the proof of Lemma 2.1 we obtain

$$\begin{aligned} \int_{\mathbb{R}^{n-1}} f_{-\epsilon} * k_\epsilon^\Pi d(\mu_{\tau, T} - \nu_{\tau, T}) - \int_{\mathbb{R}^{n-1}} (f_{+2\epsilon} - f_{-2\epsilon}) d\nu_{\tau, T} \\ \leq \int_{\mathbb{R}^{n-1}} f d(\mu_{\tau, T} - \nu_{\tau, T}) \\ \leq \int_{\mathbb{R}^{n-1}} f_{+\epsilon} * k_\epsilon^\Pi d(\mu_{\tau, T} - \nu_{\tau, T}) + \int_{\mathbb{R}^{n-1}} (f_{+2\epsilon} - f_{-2\epsilon}) d\nu_{\tau, T}. \end{aligned}$$

In view of (2.5) and the definition of $S_{\pm\epsilon}^Q$ the conclusion of the Lemma follows from the previous inequality. \square

In subsection 5.1 we will obtain bounds for the smoothing errors

$$\int_{\mathbb{R}^{nd}} (\mathbb{1}_{B(1+2\tau)} - \mathbb{1}_{B(1-2\tau)}) d\nu_T \quad \text{and} \quad \int_{\mathbb{R}^{n-1}} (\mathbb{1}_{I_1^{2\epsilon} \times \dots \times I_{n-1}^{2\epsilon}} - \mathbb{1}_{I_1^{-2\epsilon} \times \dots \times I_{n-1}^{-2\epsilon}}) d\nu_{\tau, T}$$

that arise from Lemmas 2.1 and 2.2.

2.2. Fourier transforms. To obtain bounds for $\left| R \left(S_{\pm\epsilon}^Q w_{\pm\tau, T} \right) \right|$ the strategy of [GM13] will be used, in particular we proceed via the Fourier transform. Numerous text books on Fourier analysis are available, for instance see [Gra08]. Let $\mathcal{S}(\mathbb{R}^d)$ denote the space of Schwartz functions on \mathbb{R}^d (see Section 2.2 of [Gra08]) and let $\mathcal{C}_0^\infty(\mathbb{R}^d)$ be smooth functions with compact support on \mathbb{R}^d . We note that $\mathcal{C}_0^\infty(\mathbb{R}^d) \subset \mathcal{S}(\mathbb{R}^d) \subset L^1(\mathbb{R}^d)$ and that $\mathcal{S}(\mathbb{R}^d)$ is invariant under the Fourier transform. For $v \in \mathbb{R}^{nd}$, let $Q_{++}(v) = \sum_{i=1}^n Q(v_i)$ and $\zeta_{\pm\tau}(v) = w_{\pm\tau}(v) \exp(Q_{++}(v))$. Note that if $f \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ and $g = \mathbb{1}_A$, where A is a compact subset of \mathbb{R}^d then $g * f \in \mathcal{C}_0^\infty(\mathbb{R}^d)$. This fact implies that $S_\epsilon \in \mathcal{S}(\mathbb{R}^{n-1})$ and $\zeta_{\pm\tau} \in \mathcal{S}(\mathbb{R}^{nd})$, therefore it is possible to use the Fourier inversion formula. Hence

$$(2.6) \quad S_{\pm\epsilon}^Q(v) = (2\pi)^{1-n} \int_{\mathbb{R}^{n-1}} \widehat{S}_{\pm\epsilon}(\omega) \exp(i \langle (Q(v_1) - Q(v_2), \dots, Q(v_{n-1}) - Q(v_n)), \omega \rangle) d\omega$$

and

$$(2.7) \quad \zeta_{\pm\tau}(v) = (2\pi)^{-nd} \int_{\mathbb{R}^{nd}} \widehat{\zeta}_{\pm\tau}(y) \exp(i \langle v, y \rangle) dy.$$

Therefore, by using the definition (2.3) and (2.6) we obtain

$$(2.8) \quad R\left(S_{\pm\epsilon}^Q w_{\pm\tau, T}\right) = (2\pi)^{1-n} \int_{\mathbb{R}^{n-1}} R(e_{Q, \omega} w_{\pm\tau, T}) \widehat{S}_{\pm\epsilon}(\omega) d\omega,$$

where $e_{Q, \omega}(v) = \exp(i \langle (Q(v_1) - Q(v_2), \dots, Q(v_{n-1}) - Q(v_n)), \omega \rangle)$. Also using the definitions of $w_{\pm\tau, T}$ and $\zeta_{\pm\tau}$ we have

$$R(e_{Q, \omega} w_{\pm\tau, T}) = R(e_{Q, \omega}(v) \zeta_{\pm\tau}(v/T) \exp(-Q_{++}(v/T))).$$

Combining this with (2.7) gives

$$\begin{aligned} R(e_{Q, \omega} w_{\pm\tau, T}) &= (2\pi)^{-nd} \int_{\mathbb{R}^{nd}} R(e_{Q, \omega}(v) \exp(2\pi i \langle v/T, y \rangle - Q_{++}(v/T))) \widehat{\zeta}_{\pm\tau}(y) dy \\ &= (2\pi)^{-nd} \int_{\mathbb{R}^{nd}} R(e_{Q, \omega} \tilde{e}_{Q, T, y}) \widehat{\zeta}_{\pm\tau}(y) dy \end{aligned}$$

where $\tilde{e}_{Q, T, y}(v) = \exp(i \langle v/T, y \rangle - Q_{++}(v/T))$. We now write

$$R(e_{Q, \omega} \tilde{e}_{Q, T, y}) = \theta_{T, y}(\omega) - \vartheta_{T, y}(\omega),$$

where $\theta_{T, y}(\omega)$ and $\vartheta_{T, y}(\omega)$ are defined as follows: For $x \in \mathbb{R}$, let $Q_{T, y}(x, v_i) = i(xQ(v_i) + \langle v_i/T, y \rangle) - T^{-2}Q_+(v_i)$ and for $\omega \in \mathbb{R}^{n-1}$ define

$$\overline{Q}_{T, y}(\omega, v) = \sum_{i=1}^n Q_{T, y}(\omega_i - \omega_{i-1}, v_i),$$

where $\omega_0 = \omega_n = 0$.

Remark 2.3. For the rest of the paper the convention that $\omega_0 = \omega_n = 0$, will be used in order to simplify the notation.

For $\omega \in \mathbb{R}^{n-1}$ and $y \in \mathbb{R}^{nd}$, let

$$(2.9) \quad \theta_{T, y}(\omega) = \sum_{v \in \mathbb{Z}^{nd}} \exp(\overline{Q}_{T, y}(\omega, v)) \quad \text{and} \quad \vartheta_{T, y}(\omega) = \int_{\mathbb{R}^{nd}} \exp(\overline{Q}_{T, y}(\omega, v)) dv,$$

From (2.9) and the definition of $\overline{Q}_{T, y}$, it follows that

$$(2.10) \quad \theta_{T, y}(\omega) = \prod_{i=1}^n \sum_{v_i \in \mathbb{Z}^d} \exp(Q_{T, y}(\omega_i - \omega_{i-1}, v_i))$$

and

$$(2.11) \quad \vartheta_{T, y}(\omega) = \prod_{i=1}^n \int_{\mathbb{R}^d} \exp(Q_{T, y}(\omega_i - \omega_{i-1}, v_i)) dv_i.$$

Next we define a certain bounded region of \mathbb{R}^{n-1} . Let

$$\mathcal{B}(T) = \{\omega \in \mathbb{R}^{n-1} : |\omega_i - \omega_{i-1}| \leq T^{-1} \text{ for } 1 \leq i \leq n\}.$$

Decomposing the integral over \mathbb{R}^{n-1} in (2.8) into regions we see that

$$(2.12) \quad \left| R\left(S_{\pm\epsilon}^Q w_{\pm\tau, T}\right) \right| \ll E_0(\pm\tau, \pm\epsilon, T) + E_1(\pm\tau, \pm\epsilon, T) + E_2(\pm\tau, \pm\epsilon, T)$$

where

$$\begin{aligned} E_0(\pm\tau, \pm\epsilon, T) &= \left| \int_{\mathbb{R}^{n-1} \setminus \mathcal{B}(T)} \widehat{S}_{\pm\epsilon}(\omega) \int_{\mathbb{R}^{nd}} \vartheta_{T, y}(\omega) \widehat{\zeta}_{\pm\tau}(y) dy d\omega \right| \\ E_1(\pm\tau, \pm\epsilon, T) &= \left| \int_{\mathcal{B}(T)} R(e_{Q, t} w_{\pm\tau, T}) \widehat{S}_{\pm\epsilon}(\omega) d\omega \right| \\ E_2(\pm\tau, \pm\epsilon, T) &= \left| \int_{\mathbb{R}^{n-1} \setminus \mathcal{B}(T)} \widehat{S}_{\pm\epsilon}(\omega) \int_{\mathbb{R}^{nd}} \theta_{T, y}(\omega) \widehat{\zeta}_{\pm\tau}(y) dy d\omega \right|. \end{aligned}$$

3. BOUNDING THE INTEGRALS

In this section we obtain bounds for the integrals E_0 , E_1 and E_2 , in terms of $\|\widehat{\zeta}_\tau\|_1$. Precise bounds for $\|\widehat{\zeta}_\tau\|_1$ are given in subsection 5.2. We will only consider the case when $\epsilon > 0$ and $\tau > 0$ since the other three cases can be dealt with in an identical manner. The following Theorem will be proved in Propositions 3.4, 3.7 and 3.13.

Theorem 3.1. *For all $0 < \epsilon < 1$ and $\tau > 0$, there exists $T_0 > 0$ such that for all $T > T_0$,*

$$\begin{aligned} E_0(\tau, \epsilon, T) &\ll \|\widehat{\zeta}_\tau\|_1 T^{(n-1)d-2(n-2)} \\ E_1(\tau, \epsilon, T) &\ll \left(\|\widehat{\zeta}_\tau\|_1 + \tau^{(1-nd)/2} \right) T^{(n-1)d-n+1} \\ E_2(\tau, \epsilon, T) &\ll \|\widehat{\zeta}_\tau\|_1 \mathcal{A}_\epsilon(T) T^{nd-2(n-1)}, \end{aligned}$$

where for any fixed $\epsilon > 0$ we have $\lim_{T \rightarrow \infty} \mathcal{A}_\epsilon(T) = 0$ provided that Q is irrational. (See (3.28) for a precise definition of $\mathcal{A}_\epsilon(T)$.)

The bounds for E_0 and E_1 contribute only to lower order terms. Note that the bound for E_0 is of smaller order of magnitude than $T^{nd-2(n-1)}$ all $n \in \mathbb{N}$. The bound for E_1 is of smaller order of magnitude than $T^{nd-2(n-1)}$ only for $n \leq d$. Using the fact that $\vartheta_y(t)$ can be split as in (2.11) the required bound for E_0 is relatively simple to obtain. The bound for E_1 is slightly more involved since the formula (3.9) is used. This means that, although one can still take advantage of the splitting given in (2.10), the sums in the product may include 0 and this causes extra difficulties. To overcome these difficulties we employ Lemma 3.6, which enables us to bound the minimum of certain quantities from below by a weighted average. The bound for E_2 contributes to the main term and this term depends on the arithmetic properties of Q . Using (2.10), the bound for E_2 follows reasonably directly from results in [GM13].

3.1. Bound for E_0 . The following bound will be used in subsections 3.1 and 3.2 to obtain bounds for E_0 and E_1 . It is relatively straightforward to prove via a direct computation involving Gaussian integrals (see Formula (3.28) in [GM13]). The notation Q_+^{-1} will stand for the positive definite quadratic form that corresponds to the matrix Q_+^{-1} .

Lemma 3.2. *For $1 \leq i \leq n-1$, all $y \in \mathbb{R}^{nd}$, $x \in \mathbb{R}$ and $T > 0$,*

$$\left| \int_{\mathbb{R}^d} \exp(Q_{T,y}(x, v_i)) dv_i \right| \ll g_T(x) T^{d/2} \exp\left(-\frac{1}{4} g_T(x) Q_+^{-1}(y_i/T)\right),$$

where $g_T(x) = T/\sqrt{1 + (xT^2)^2}$.

We will need estimates for the Fourier transform of the smoothened characteristic function.

Lemma 3.3. *For all $\epsilon > 0$ and $\omega \in \mathbb{R}^{n-1}$,*

$$|\widehat{S}_\epsilon(\omega)| \ll \prod_{i=1}^{n-1} \min\left\{1, \left|\frac{1}{\omega_i}\right|\right\} \exp\left(-c\sqrt{\epsilon|\omega_i|}\right).$$

Proof. Using the definition of S_ϵ we get $|\widehat{S}_\epsilon(\omega)| \leq |\widehat{1}_{I_1^\epsilon \times \dots \times I_{n-1}^\epsilon}(\omega)| |\widehat{k}_\epsilon^1(\omega_1) \dots \widehat{k}_\epsilon^1(\omega_{n-1})|$. Then a simple computation and (2.1) gives

$$|\widehat{S}_\epsilon(\omega)| \ll \prod_{i=1}^{n-1} \left| \frac{1}{\omega_i} \sin(\omega_i \pi |I_i|) \right| \exp\left(-c\sqrt{\epsilon|\omega_i|}\right).$$

Since $\left| \frac{\sin(kx)}{x} \right| \leq \min\{k, |1/x|\}$ for all $x \in \mathbb{R}$, the claim of the Lemma follows. \square

The bound for E_0 is obtained by using Lemmas 3.2 and 3.3, together with some elementary estimates of integrals of powers of $g_T(x)$. For $\omega \in \mathbb{R}^{n-1}$, define $G_T(\omega) = \prod_{i=1}^n g_T(\omega_i - \omega_{i-1})$.

Proposition 3.4. *For all $\epsilon > 0$, $\tau > 0$ and $T > 0$,*

$$E_0(\tau, \epsilon, T) \ll \|\widehat{\zeta}_\tau\|_1 T^{(n-1)d-2(n-2)}.$$

Proof. Using (2.11) and Lemma 3.2,

$$|\vartheta_{T,y}(\omega)| \ll T^{nd/2} G_T(\omega)^{d/2}.$$

Note that for any $x \in \mathbb{R}$ and $T \in \mathbb{R}$, $g_T(x) \leq |Tx|^{-1}$. It follows that for all $\omega \in \mathbb{R}^{n-1}$ and $y \in \mathbb{R}^{nd}$ and $1 \leq i, j \leq n$ with $i \neq j$,

$$(3.1) \quad |\vartheta_{T,y}(\omega)| \ll T^{(n-2)d/2} \left(\frac{G_T(\omega)}{g_T(\omega_i - \omega_{i-1}) g_T(\omega_j - \omega_{j-1})} \right)^{d/2} (|\omega_i - \omega_{i-1}| |\omega_j - \omega_{j-1}|)^{-d/2}.$$

Choose $1 \leq l \leq n$ with $l \neq i$ or j . Since, we assume that $n \geq 3$, this is always possible. Note that $g_T(x) \leq T$ and hence

$$(3.2) \quad \left(\frac{G_T(\omega)}{g_T(\omega_i - \omega_{i-1}) g_T(\omega_j - \omega_{j-1})} \right)^{d/2} \leq T^{d/2} \tilde{G}_T(\omega)^{d/2},$$

where

$$\tilde{G}_T(\omega) = \frac{G_T(\omega)}{g_T(\omega_i - \omega_{i-1}) g_T(\omega_j - \omega_{j-1}) g_T(\omega_l - \omega_{l-1})}.$$

Let $T_n = (n-1)T$ and

$$\mathcal{B}_1 = \bigcup_{\substack{1 \leq i, j \leq n \\ i \neq j}} \mathcal{B}_{i,j}, \text{ where } \mathcal{B}_{i,j} = \{\omega \in \mathbb{R}^{n-1} : |\omega_i - \omega_{i-1}| > T_n^{-1}, |\omega_j - \omega_{j-1}| > T_n^{-1}\}.$$

Note that $\mathbb{R}^{n-1} \setminus \mathcal{B}(T) \subset \mathcal{B}_1$. (Or, equivalently, $\mathbb{R}^{n-1} \setminus \mathcal{B}_1 \subset \mathcal{B}(T)$.) To see this, suppose that $\omega \in \mathbb{R}^{n-1} \setminus \mathcal{B}_1$, then $|\omega_i - \omega_{i-1}| > T_n^{-1}$ for at most one $1 \leq i \leq n$. If $|\omega_i - \omega_{i-1}| \leq T_n^{-1}$ for all $1 \leq i \leq n$, then clearly $\omega \in \mathcal{B}(T)$. Suppose $1 \leq l \leq n$ is such that $|\omega_l - \omega_{l-1}| > T_n^{-1}$. Now note that $|\omega_l - \omega_{l-1}| = \left| \sum_{1 \leq i \leq n, i \neq l} (\omega_i - \omega_{i-1}) \right| \leq T^{-1}$ by the triangle inequality and hence $\omega \in \mathcal{B}(T)$.

Using the definition of E_0 ,

$$(3.3) \quad \begin{aligned} E_0(\tau, \epsilon, T) &= \left| \int_{\mathbb{R}^{n-1} \setminus \mathcal{B}(T)} \hat{S}_\epsilon(\omega) \int_{\mathbb{R}^{nd}} \vartheta_{T,y}(\omega) \hat{\zeta}_\tau(y) dy d\omega \right| \\ &\leq \int_{\mathbb{R}^{n-1} \setminus \mathcal{B}(T)} \left| \hat{S}_\epsilon(\omega) \int_{\mathbb{R}^{nd}} \vartheta_{T,y}(\omega) \hat{\zeta}_\tau(y) dy \right| d\omega \\ &\leq \sum_{\substack{1 \leq i, j \leq n \\ i \neq j}} \int_{\mathcal{B}_{i,j}} \left| \hat{S}_\epsilon(\omega) \int_{\mathbb{R}^{nd}} \vartheta_{T,y}(\omega) \hat{\zeta}_\tau(y) dy \right| d\omega. \end{aligned}$$

From Lemma 3.3 we get that for all $\epsilon > 0$ we have the uniform bound, $|\hat{S}_\epsilon(\omega)| \ll 1$. Hence using (3.1) and (3.2),

$$(3.4) \quad \int_{\mathcal{B}_{i,j}} \left| \hat{S}_\epsilon(\omega) \int_{\mathbb{R}^{nd}} \vartheta_{T,y}(\omega) \hat{\zeta}_\tau(y) dy \right| d\omega \ll \|\hat{\zeta}_\tau\|_1 T^{(n-1)d/2} \int_{\mathcal{B}_{i,j}} \tilde{G}_T(\omega)^{d/2} (|\omega_i - \omega_{i-1}| |\omega_j - \omega_{j-1}|)^{-d/2} d\omega.$$

By doing the change of variables $\varphi : \omega_i - \omega_{i-1} \rightarrow \begin{cases} \xi_i & \text{if } i < l \\ \xi_{i-1} & \text{if } i > l \end{cases}$ we get

$$(3.5) \quad \int_{\mathcal{B}_{i,j}} \tilde{G}_T(\omega)^{d/2} (|\omega_i - \omega_{i-1}| |\omega_j - \omega_{j-1}|)^{-d/2} d\omega \leq \int_{\varphi(\mathcal{B}_{i,j})} \prod_{\substack{1 \leq k \leq n-1 \\ k \neq i, j}} g_T(\xi_k)^{d/2} (|\xi_i| |\xi_j|)^{-d/2} d\xi,$$

where $\varphi(\mathcal{B}_{i,j}) = \{\xi \in \mathbb{R}^{n-1} : |\xi_i| \geq T_n^{-1}, |\xi_j| \geq T_n^{-1}\}$. Note that

$$\int_{T_n^{-1}}^{\infty} x^{-d/2} dx \ll T^{d/2-1}.$$

By making change of variables $T^2 x = \sinh y$, we get

$$\int_{\mathbb{R}} g_T(x)^{d/2} dx = T^{d/2-2} \int_{\mathbb{R}} \frac{1}{\cosh^{d/4+1} y} dy \ll T^{d/2-2}.$$

The last two observations, (3.4) and (3.5) imply that for all $1 \leq i, j \leq n$ with $i \neq j$,

$$\int_{\mathcal{B}_{i,j}} \left| \hat{S}_\epsilon(\omega) \int_{\mathbb{R}^{nd}} \vartheta_{T,y}(\omega) \hat{\zeta}_\tau(y) dy \right| d\omega \leq \|\hat{\zeta}_\tau\|_1 T^{(n-1)d-2(n-2)}.$$

The conclusion of the Lemma follows from (3.3). \square

3.2. Bound for E_1 . We will need two preliminary Lemmas. The first is probably standard, but for completeness, a proof is provided.

Lemma 3.5. *For any $c > 0$, there exists a positive constant B such that, for any $y \in \mathbb{R}^{nd}$*

$$\sum_{m \in \mathbb{Z}^{nd} \setminus \{0\}} \exp(-c \|y - m\|^2) < B.$$

Proof. For $v \in [-1/2, 1/2]^{nd}$, $\|v\|^2 \leq nd/4$. Hence

$$\int_{[-1/2, 1/2]^{nd}} \exp(-c \|u + v\|^2) dv \geq \exp(-c \|u\|^2 - ndc/4) \int_{[-1/2, 1/2]^{nd}} \exp(-2c \langle u, v \rangle) dv.$$

It is easy to check that for any $u \in \mathbb{R}^{nd}$, $\int_{[-1/2, 1/2]^{nd}} \exp(-2c \langle u, v \rangle) dv \geq 1$ and hence we get the inequality

$$\exp(-c \|u\|^2) \ll \int_{[-1/2, 1/2]^{nd}} \exp(-c \|u + v\|^2) dv.$$

Hence

$$\begin{aligned} \sum_{m \in \mathbb{Z}^{nd} \setminus \{0\}} \exp(-c \|y - m\|^2) &\ll \sum_{m \in \mathbb{Z}^{nd} \setminus \{0\}} \int_{[-1/2, 1/2]^{nd}} \exp(-c \|y - m + v\|^2) dv \\ &= \sum_{m \in \mathbb{Z}^{nd} \setminus \{0\}} \int_{[-1/2, 1/2]^{nd} + y - m} \exp(-c \|z\|^2) dz \\ &\ll \int_{\mathbb{R}^{nd}} \exp(-c \|z\|^2) dz < \infty. \end{aligned}$$

□

The second preliminary Lemma is the crucial step in obtaining the estimate for E_1 . For $\omega \in \mathbb{R}^{n-1}$, let $G_T^*(\omega) = \min_{1 \leq i \leq n} g_T(\omega_i - \omega_{i-1})^2$.

Lemma 3.6. *Let $\omega \in \mathcal{B}(T)$ and $\{s_1, \dots, s_n\} = \{\omega_i - \omega_{i-1}\}_{1 \leq i \leq n}$. Suppose $s_1 = \max_{1 \leq i \leq n} |s_i|$. Then, there exist positive constants c_1, \dots, c_n , and b_2, \dots, b_n such that, for all $T \geq 1$,*

$$G_T^*(\omega) \geq \sum_{i=1}^n \frac{c_i}{T^{b_i}} g_T(s_i)^2,$$

where $b_1 = 0$ and $\sum_{i=2}^n b_i \leq 2(n-2)$.

Proof. For $\omega \in \mathcal{B}(T)$, let $\{s_1, \dots, s_n\} = \{\omega_i - \omega_{i-1}\}_{1 \leq i \leq n}$ and $s_1 = \max_{1 \leq i \leq n} |s_i|$. Thus, $G_T^*(\omega) = g_T(s_0)^2$. Let

$$\alpha_i = \begin{cases} -\log |s_i| / \log T & \text{if } |s_i| \in [1/T^2, 1/T] \\ 2 & \text{if } |s_i| \in [0, 1/T^2]. \end{cases}$$

Note that $1 \leq \alpha_i \leq 2$ for all $1 \leq i \leq n$ and $\alpha_1 = \min_{1 \leq i \leq n} \alpha_i$. For all s_i such that $|s_i| \in [1/T^2, 1/T]$ we have $|s_i| = 1/T^{\alpha_i}$. Thus

$$(3.6) \quad \left(\frac{g_T(s_0)}{g_T(s_i)} \right)^2 = \frac{1 + T^{4-2\alpha_i}}{1 + T^{4-2\alpha_0}} \geq \frac{1}{2T^{2(\alpha_i - \alpha_1)}}.$$

For all s_i such that $|s_i| \in [0, 1/T^2]$ we have $g_T(s_i)^2 \in [T^2/2, T^2]$. Thus

$$(3.7) \quad \left(\frac{g_T(s_1)}{g_T(s_i)} \right)^2 \geq \frac{1}{1 + T^{4-2\alpha_1}} \geq \frac{1}{2T^{2(\alpha_i - \alpha_1)}}.$$

Let $\delta_i = \alpha_i - \alpha_1$. Note that for all $1 \leq i \leq n-1$, $0 \leq \delta_i \leq 1$. Moreover, $\sum_{i=1}^n s_i = 0$ and hence

$$|s_1| \leq \sum_{s_i \in \{s_1, \dots, s_n\} \setminus \{s_1\}} |s_i|,$$

which implies that

$$\frac{1}{T^{\alpha_1}} \leq \sum_{i=2}^{n-1} \frac{1}{T^{\alpha_i}} = \sum_{i=2}^{n-1} \frac{1}{T^{\delta_i + \alpha_1}}.$$

Thus

$$(3.8) \quad 1 \leq \sum_{i=2}^{n-1} \frac{1}{T^{\delta_i}}.$$

Let $\sum_{i=2}^{n-1} \delta_i = \Delta$ and note that for all $2 \leq i \leq n-1$, $\Delta - \delta_i \leq n-2$. Hence by multiplying by the denominators in (3.8) we get

$$T^\Delta \leq \sum_{i=2}^{n-1} T^{\Delta - \delta_i} \leq (n-2) T^{n-2}.$$

Since this holds for all $T \geq 1$ we get $\sum_{i=2}^{n-1} \delta_i \leq n-2$. Finally, note that from (3.6) and (3.7), we obtain

$$G_T^*(\omega) = g_T(s_0)^2 \geq \frac{1}{n} \left(g_T(s_1)^2 + \sum_{s_i \in \{s_1, \dots, s_n\} \setminus \{s_1\}} \frac{1}{2T^{2(\alpha_i - \alpha_1)}} g_T(s_i)^2 \right),$$

since $\sum_{s_i \in \{s_1, \dots, s_n\} \setminus \{s_1\}} 2(\alpha_i - \alpha_1) = \sum_{i=2}^{n-1} 2\delta_i \leq 2(n-2)$ the claim of the Lemma follows. \square

Using the preceding two results we can now obtain a bound for E_1 . This is done by using Poisson summation to convert the difference $\theta_{T,y}(\omega) - \vartheta_{T,y}(\omega)$, into a sum over $m \in \mathbb{Z}^{nd} \setminus \{0\}$. An estimate for each term in the sum can be obtained provided that $\|y/T - m\|$ is bounded from below. The integral over $y \in \mathbb{R}^{nd}$ is then decomposed into boxes centred at each point of \mathbb{Z}^{nd} . Thus, the estimates for the summands can be used and there will be additional term coming from the point at the centre of the box under consideration. Lemma 3.6 is used to bound a function of the form $G_T(\omega)^{d/2} \exp(-G_T^*(\omega))$. Finally, Lemma 5.8 is used to estimate the term that arises for small $\|y/T - m\|$.

Proposition 3.7. *For all $\epsilon > 0$, $\tau > 0$ and $T \geq \tau^{-1}$,*

$$E_1(\tau, \epsilon, T) \ll \left(\|\widehat{\zeta}_\tau\|_1 + \tau^{(1-nd)/2} \right) T^{(n-1)d-n+1}.$$

Proof. Recall (see Section 2)

$$E_1(\tau, \epsilon, T) = \left| \int_{\mathcal{B}(T)} R(e_{Q,t} w_{\tau,T}) \widehat{S}_\epsilon(\omega) d\omega \right|$$

and

$$R(e_{Q,\omega} w_{\tau,T}) = \int_{\mathbb{R}^{nd}} R(e_{Q,\omega} \tilde{e}_{Q,T,y}) \widehat{\zeta}_\tau(y) dy = \int_{\mathbb{R}^{nd}} (\theta_{T,y}(\omega) - \vartheta_{T,y}(\omega)) \widehat{\zeta}_\tau(y) dy.$$

Note that $e_{Q,\omega} \tilde{e}_{Q,T,y} \in \mathcal{S}(\mathbb{R}^{nd})$ and thus, there exists a constant C , so that $|e_{Q,\omega} \tilde{e}_{Q,T,y}(v)| + |\widehat{e_{Q,\omega} \tilde{e}_{Q,T,y}}(v)| \leq C(1 + \|x\|)^{-(n+1)}$. Hence, using Poisson summation (Theorem 3.1.17 in [Gra08])

$$(3.9) \quad \theta_{T,y}(\omega) - \vartheta_{T,y}(\omega) = \sum_{m \in \mathbb{Z}^{nd} \setminus \{0\}} \vartheta_{T,y-Tm}(\omega).$$

By using (3.9),

$$(3.10) \quad E_1(\tau, \epsilon, T) = \left| \int_{\mathcal{B}(T)} \widehat{S}_\epsilon(\omega) \int_{\mathbb{R}^{nd}} \sum_{m \in \mathbb{Z}^{nd} \setminus \{0\}} \vartheta_{T,y-Tm}(\omega) \widehat{\zeta}_\tau(y) dy d\omega \right|.$$

Let $\Sigma_{T,\omega,y} = \sum_{m \in \mathbb{Z}^{nd} \setminus \{0\}} \vartheta_{T,y-Tm}(\omega)$. By Lemma 3.2 we have

$$|\Sigma_{T,\omega,y}| \ll \sum_{m \in \mathbb{Z}^{nd} \setminus \{0\}} |\vartheta_{y-Tm}(\omega)| \ll T^{nd/2} G_T(\omega)^{d/2} \sum_{m \in \mathbb{Z}^{nd} \setminus \{0\}} \exp(-Q_{T,\omega}^*(y, m)),$$

where

$$Q_{T,\omega}^*(y, m) = \frac{1}{4} \sum_{1 \leq i \leq n} g_T(\omega_i - \omega_{i-1})^2 Q_+^{-1}(y_i/T - m_i).$$

Note Q_+^{-1} is a positive definite quadratic form and because $\omega \in \mathcal{B}(T)$ we have (using that $T > 1$) $1/2 < g_T(\omega_i - \omega_{i-1})^2$ for $1 \leq i \leq n$. Recall, λ_{\max} is the maximum (in terms of absolute value) eigenvalue of Q . Therefore,

$\min_{\lambda \in \text{sp}(Q_+^{-1})} |\lambda| = \lambda_{\max}^{-2}$. Let $c_Q = \frac{1}{16} \lambda_{\max}^{-2}$. We get

$$\begin{aligned} Q_{T,\omega}^*(y, m) &\geq 2c_Q \sum_{1 \leq i \leq n} \|y_i/T - m_i\|^2 \\ (3.11) \quad &= 2c_Q \|y/T - m\|^2. \end{aligned}$$

Also, $Q_{T,\omega}^*(y, m) \geq \frac{1}{4} G_T^*(\omega) Q_{++}^{-1}(y/T - m)$. If $y/T \in [-1/2, 1/2]^{nd}$ then $\|y/T - m\| \geq 1/2$ for $m \in \mathbb{Z}^{nd} \setminus \{0\}$. Therefore, $Q_{T,\omega}^*(y, m) \geq c_Q G_T^*(\omega)$ and

$$(3.12) \quad \exp(-Q_{T,\omega}^*(y, m)) = \left(\exp\left(-\frac{1}{2} Q_{T,\omega}^*(y, m)\right) \right)^2 \leq \exp\left(-\frac{1}{2} Q_{T,\omega}^*(y, m)\right) \exp(-c_Q G_T^*(\omega)).$$

Thus, by combining Lemma 3.5, (3.11) and (3.12),

$$(3.13) \quad \sum_{m \in \mathbb{Z}^{nd} \setminus \{0\}} \exp(-Q_{T,\omega}^*(y, m)) \ll \exp(-c_Q G_T^*(\omega)).$$

If $y/T \in [-1/2, 1/2]^{nd} + m_0$ for some $m_0 \in \mathbb{Z}^{nd} \setminus \{0\}$ then $\|y/T - m\| \geq 1/2$ for $m \in \mathbb{Z}^{nd} \setminus \{m_0\}$. Therefore, we can repeat the above argument and get that

$$(3.14) \quad \sum_{m \in \mathbb{Z}^{nd} \setminus \{0\}} \exp(-Q_{T,\omega}^*(y, m)) \ll \exp(-c_Q G_T^*(\omega)) + \exp(-Q_{T,\omega}^*(y', 0)),$$

where $y'/T = y/T - m_0 \in [-1/2, 1/2]^{nd}$. It follows from (3.13) that

$$(3.15) \quad \int_{B_\infty(T/2)} \left| \Sigma_{T,\omega,y} \widehat{\zeta}_\tau(y) \right| dy \ll T^{nd/2} G_T(\omega)^{d/2} \exp(-c_Q G_T^*(\omega)) \|\widehat{\zeta}_\tau\|_1$$

and from (3.14) that

$$\begin{aligned} (3.16) \quad &\int_{\mathbb{R}^{nd} \setminus B_\infty(T/2)} \left| \Sigma_{T,\omega,y} \widehat{\zeta}_\tau(y) \right| dy \\ &\ll T^{nd/2} G_T(\omega)^{d/2} \left(\exp(-c_Q G_T^*(\omega)) \|\widehat{\zeta}_\tau\|_1 + \int_{\mathbb{R}^{nd} \setminus B_\infty(T/2)} \exp(-\overline{Q}_{T,\omega}(y', 0)) |\widehat{\zeta}_\tau(y)| dy \right). \end{aligned}$$

By Lemma 3.6 for $\{s_1, \dots, s_n\} = \{\omega_i - \omega_{i-1}\}_{1 \leq i \leq n}$, with $\omega \in \mathcal{B}(T)$ there exist positive constants c_1, \dots, c_n , and b_2, \dots, b_n such that

$$\exp(-c_Q G_T^*(\omega)) \leq \exp\left(-\sum_{i=1}^n \frac{c_i}{T^{b_i}} g_T(s_i)^2\right),$$

where $b_1 = 0$ and $\sum_{i=2}^n b_i \leq 2(n-2)$. Consider the function $x^{d/2} \exp(-kx^2)$, it obtains its maximum at $x = \sqrt{d/4k}$ and this maximum value is $(d/4k)^{d/4} \exp(-d/4)$. Hence

$$(3.17) \quad G_T(\omega)^{d/2} \exp(-c_Q G_T^*(\omega)) \ll T^{\frac{d}{4} \sum_{i=1}^n b_i} \leq T^{d(n-2)/2}.$$

Also note that

$$\exp(-Q_{T,\omega}^*(y', 0)) \leq \exp\left(-\frac{G_T^*(\omega)}{4} Q_{++}^{-1}(y'/T)\right)$$

and thus by using Lemma 3.6 again

$$G_T(\omega)^{d/2} \exp(-Q_{T,\omega}^*(y', 0)) \ll T^{d(n-2)/2} (Q_{++}^{-1}(y'/T))^{-nd/4}.$$

Note that $g_T(x) \leq T$ for all $x \in \mathbb{R}$ and hence we also have the trivial bound

$$G_T(\omega)^{d/2} \exp(-Q_{T,\omega}^*(y', 0)) \leq T^{nd/2}.$$

Hence

$$G_T(\omega)^{d/2} \exp(-Q_{T,\omega}^*(y', 0)) \ll \min \left\{ T^{nd/2}, T^{d(n-2)/2} (Q_{++}^{-1}(y'/T))^{-nd/4} \right\} \ll \left(\frac{Q_{++}^{-1}(y'/T)}{T^{2(n-2)/n}} + \frac{1}{T^2} \right)^{-nd/4}.$$

For $y \in \mathbb{R}^{nd}$, let $\|y\|_{\mathbb{Z}^{nd}} = \min_{z \in \mathbb{Z}^{nd}} \|z - y\|$. By rearranging and using the fact that $Q_{++}^{-1}(y'/T) \geq \frac{1}{\lambda_{\max}^2} \|y/T\|_{\mathbb{Z}^{nd}}$ we get

$$\begin{aligned}
 (3.18) \quad \int_{\mathbb{R}^{nd} \setminus B_{\infty}(T/2)} \left(\frac{Q_{++}^{-1}(y'/T)}{T^{2(n-2)/n}} + \frac{1}{T^2} \right)^{-nd/4} |\widehat{\zeta}_{\tau}(y)| dy \\
 \ll \int_{\mathbb{R}^{nd} \setminus B_{\infty}(T/2)} \left(\frac{1}{\lambda_{\max}^2 T^{2(n-2)/n}} \|y/T\|_{\mathbb{Z}^{nd}} + \frac{1}{T^2} \right)^{-nd/4} |\widehat{\zeta}_{\tau}(y)| dy \\
 = T^{d(n-2)/2} \int_{\mathbb{R}^{nd} \setminus B_{\infty}(T/2)} \frac{|\widehat{\zeta}_{\tau}(y)|}{(\lambda_{\max}^{-2} \|y/T\|_{\mathbb{Z}^{nd}} + T^{-4/n})^{nd/4}} dy.
 \end{aligned}$$

Note that

$$\begin{aligned}
 \int_{\mathbb{R}^{nd} \setminus B_{\infty}(T/2)} \frac{|\widehat{\zeta}_{\tau}(y)|}{(\lambda_{\max}^{-2} \|y/T\|_{\mathbb{Z}^{nd}} + T^{-4/n})^{nd/4}} dy &= \int_{\mathbb{R}^{nd} \setminus B_{\infty}(T/2)} T^d \frac{|\widehat{\zeta}_{\tau}(y)|}{(\lambda_{\max}^{-2} T^{4/n} \|y/T\|_{\mathbb{Z}^{nd}} + 1)^{nd/4}} dy \\
 &\ll \int_{\mathbb{R}^{nd} \setminus B_{\infty}(T/2)} T^d |\widehat{\zeta}_{\tau}(y)| dy.
 \end{aligned}$$

Hence, if $T \geq \tau^{-1}$, we can apply Lemma 5.8 and use (3.18) to get

$$\int_{\mathbb{R}^{nd} \setminus B_{\infty}(T/2)} \left(\frac{Q_{++}^{-1}(y'/T)}{T^{2(n-2)/n}} + \frac{1}{T^2} \right)^{-nd/4} |\widehat{\zeta}_{\tau}(y)| dy \ll T^{d(n-2)/2} \tau^{(1-nd)/2}.$$

Finally by using this, (3.10), (3.15), (3.16) and (3.17) we see that, provided $T \geq \tau^{-1}$,

$$E_1(\tau, \epsilon, T) \ll T^{nd-d} \left(\|\widehat{\zeta}_{\tau}\|_1 + \tau^{(1-nd)/2} \right) \int_{B(T)} |\widehat{S}_{\epsilon}(\omega)| d\omega \ll T^{nd-d-n+1} \left(\|\widehat{\zeta}_{\tau}\|_1 + \tau^{(1-nd)/2} \right),$$

since Lemma 3.3 implies that $\int_{B(T)} |\widehat{S}_{\epsilon}(\omega)| d\omega \ll T^{1-n}$. □

3.3. Bound for E_2 . This term contributes the main error term. It is easy to see that

$$(3.19) \quad E_2(\tau, \epsilon, T) \ll \|\widehat{\zeta}_{\tau}\|_1 \int_{\mathbb{R}^{n-1} \setminus B(T)} |\widehat{S}_{\epsilon}(\omega)| \sup_{y \in \mathbb{R}^{nd}} |\theta_{T,y}(\omega)| d\omega.$$

For $x \in \mathbb{R}$, let

$$\psi(T, x) = \sum_{(m, \bar{m}) \in \mathbb{Z}^d \times \mathbb{Z}^d} \exp(-H_{T,x}(m, \bar{m})),$$

where

$$H_{T,x}(m, \bar{m}) = T^2 Q_+^{-1} \left(m - \frac{2}{\pi} x Q \bar{m} \right) + T^{-2} Q_+(\bar{m})$$

is a positive definite quadratic form on \mathbb{Z}^{2d} . From Lemma 3.3 of [GM13] and (2.10) we have that

$$(3.20) \quad |\theta_{T,y}(\omega)| \ll T^{nd/2} \prod_{i=1}^n \sqrt{\psi(T, \omega_i - \omega_{i-1})}.$$

We can now use results and the strategy of [GM13]. Namely, the quadratic form $H_{T,x}$ is parametrised in terms of the action of certain geodesic and unipotent elements of $SL_2(\mathbb{R})$. Write $v \in \mathbb{R}^{2d}$ as $v = (v_1, \dots, v_d)$, where $v_i \in \mathbb{R}^2$. Consider the action of $SL_2(\mathbb{R})$ on \mathbb{R}^{2d} given by $gv = (gv_1, \dots, gv_d)$. This is the action studied in [GM13], Section 4. For $T > 0$, let $d_T = \begin{pmatrix} T & 0 \\ 0 & T^{-1} \end{pmatrix}$ and for $x \in \mathbb{R}$, let $u_x = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$. For $(m, \bar{m}) \in \mathbb{Z}^d \times \mathbb{Z}^d$, let $m' = Q_+^{-1/2} m$ and $\bar{m}' = Q_+^{-1/2} Q \bar{m}$. Let $\Pi \in SL_{2d}(\mathbb{Z})$ be the permutation matrix such that

$$\Pi(m', \bar{m}') = (m'_1, \bar{m}'_1, m'_2, \bar{m}'_2, \dots, m'_d, \bar{m}'_d).$$

Let $\Lambda_Q = \Pi \begin{pmatrix} Q_+^{-1/2} & 0 \\ 0 & Q_+^{-1/2} Q \end{pmatrix} \mathbb{Z}^{2d}$. It is shown in [GM13] (equation (4.21)) and is indeed not hard to see, that

$$(3.21) \quad H_{T,x}(m, \bar{m}) = \left\| d_T u_{\frac{x}{2}} \Pi(m', \bar{m}') \right\|^2.$$

The following Lemma follows from Lemma 3.4 of [GM13].

Lemma 3.8. *Let Δ be a lattice in \mathbb{R}^d . Then*

$$\sum_{v \in \Delta} \exp(-\|v\|^2) \ll |\{v \in \Delta : \|v\|_\infty < 1\}|,$$

where the implicit constant does not depend on Δ .

It is easy to see that by using (3.21) and the definition of ψ that Lemma 3.8 implies

$$(3.22) \quad \psi(T, x) \ll \left| \left\{ v \in d_T u_{\frac{2}{\pi}x} \Lambda_Q : \|v\|_\infty < 1 \right\} \right|.$$

Next we introduce the function α on the space of lattices. For more details see Section 4 of [GM13]. Let $\Delta \in \mathbb{R}^{2d}$ be a lattice. Let U be a subspace of \mathbb{R}^{2d} , we say that U is Δ -rational if $U \cap \Delta$ is a lattice in U . For $1 \leq i \leq 2d$ and a quasinorm, $|\cdot|_i$ on $\bigwedge^i(\mathbb{R}^{2d})$ we define $d_\Delta(U) = |u_1 \wedge \cdots \wedge u_i|_i$ where u_1, \dots, u_i is a basis of $U \cap \Delta$ over \mathbb{Z} . Note that $d_\Delta(U)$ does not depend on the choice of basis. Note that any two quasinorms on \mathbb{R}^{2d} are equivalent. For details about the precise choice of quasinorm, see section 5 of [GM13]. Let

$$\Psi_i(\Delta) = \{U : U \text{ is a } \Delta\text{-rational subspace of } \mathbb{R}^{2d} \text{ with } \dim U = i\}.$$

Define

$$\alpha_i(\Delta) = \sup_{U \in \Psi_i(\Delta)} \frac{1}{d_\Delta(U)} \quad \text{and} \quad \alpha(\Delta) = \max_{1 \leq i \leq 2d} \alpha_i(\Delta).$$

The following Lemma collects together several results from [GM13] regarding the alpha functions.

Lemma 3.9. *Let $\Lambda_{T,x} = d_T u_x \Lambda_Q$ then*

- (i) *For any $x \in \mathbb{R}$ and $\mu > 1$, $|\{v \in \Lambda_{T,x} : \|v\| \leq \mu\}| \asymp \mu^{2d} \alpha_d(\Lambda_{T,x})$.*
- (ii) *For any $x \in \mathbb{R}$, $\alpha(\Lambda_{T,x}) \asymp \alpha_d(\Lambda_{T,x})$.*
- (iii) *If $T \geq \lambda_{\max}$, then $\sup_{x \in \mathbb{R}} \alpha_d(\Lambda_{T,x}) \ll T^d$.*
- (iv) *If $|x| \geq (\lambda_{\min} T)^{-1}$, then $\alpha_d(\Lambda_{T,x}) \ll (T^{-1} + |xT|)^d$.*
- (v) *Let $I = [a, b]$ with $a \in (0, 1)$ and $b > 2$. If Q is irrational then $\lim_{T \rightarrow \infty} (\sup_{x \in I} \alpha_d(\Lambda_{T,x}) T^{-d}) = 0$.*
- (vi) *Let $I = [a, b]$ with $a \in (0, 1)$ and $b > 2$. If Q is of Diophantine type (κ, A) then $\sup_{x \in I} \alpha_d(\Lambda_{T,x}) T^{-d} \ll \max \{a^{-(\kappa+1)}, b^\kappa\} T^{2(\kappa-1)}$.*

These facts are all proved in Section 4 of [GM13]. For (i) see Lemma 4.6. For (ii) see corollary 4.7. For (iii) and (iv) see Lemma 4.8. For (v) and (vi) see Corollary 4.11.

We proceed by finding an approximation of the integral in (3.19) over a fixed box in \mathbb{R}^{n-1} . As in [GM13] we use (3.20) together with the parametrisation of the quadratic form $H_{T,x}$ discussed previously. The function α is then introduced via (3.22) and Lemma 3.9, part (i). Let $K = SO_2(\mathbb{R})$. A change of variables is used to convert the problem from an integral over a box in \mathbb{R}^{n-1} into an integral over K^{n-1} . This is done via Lemma 4.9 of [GM13], which is reproduced below.

Lemma 3.10. *Let $x \in [-2, 2]$, $T > 1$ and Δ be a lattice in \mathbb{R}^{2d} , then for all $1 \leq i \leq 2d$,*

$$\alpha_i(d_T u_x \Delta) \ll \alpha_i(d_T k_\theta \Delta),$$

where $k_\theta = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \in SO_2(\mathbb{R})$ and $x = \tan \theta$.

Then the problem is to understand the averages over translated K orbits. This problem is studied in Section 5 of [GM13]. In particular, the following Theorem (Theorem 5.11 in [GM13]) is the critical estimate.

Theorem 3.11. *Let $2/d < \beta$. Then there exists a constant C depending only on β and the choice of quasinorms used to define α , such that for all $g \in SL_2(\mathbb{R})$ and any lattice $\Delta \in \mathbb{R}^{2d}$,*

$$\int_K \alpha(gk\Delta)^\beta d\nu(k) \leq C \alpha(\Delta)^\beta \|g\|^{\beta d - 2}.$$

As previously remarked the bounds for E_2 contain the arithmetic information regarding the quadratic form Q . This is encoded via the following function. For an interval I , and $2/d < \beta < 1/2$ define $\gamma_{I,\beta}(T) = \sup_{x \in I} (T^{-d} \alpha_d(\Lambda_{T,x}))^{1/2-\beta}$. Let $L_0 = L_n$ be the empty set, L_1, \dots, L_{n-1} be intervals and $\Gamma_{L_1, \dots, L_{n-1}, \beta}(T) = \prod_{i=1}^n \gamma_{L_i - L_{i-1}, \beta}(T)$ where the notation $L_i - L_j = \{l_i - l_j : l_i \in L_i, l_j \in L_j\}$.

Lemma 3.12. *Let L_1, \dots, L_{n-1} be closed intervals of length 2, $L = L_1 \times \cdots \times L_{n-1}$ and $S_\epsilon^*(L) = \sup_{\omega \in L} |\widehat{S}_\epsilon(\omega)|$. Then for all $\epsilon > 0$ and $T \geq \lambda_{\max}$,*

$$\int_L |\widehat{S}_\epsilon(\omega)| \sup_{y \in \mathbb{R}^{nd}} |\theta_{T,y}(\omega)| d\omega \ll S_\epsilon^*(L) \Gamma_{L_1, \dots, L_{n-1}, \beta}(T) T^{nd-2(n-1)}.$$

Proof. By using (3.22) and Lemma 3.9, part (i) we get

$$\psi(T, x) \ll \left| \left\{ v \in \Lambda_{T, \frac{2}{\pi}x} : \|v\| \leq d^{1/2} \right\} \right| \ll \alpha_d \left(\Lambda_{T, \frac{2}{\pi}x} \right).$$

It easily follows from the previous formula and the definition of $\gamma_{I, \beta}(T)$ that for all $x \in \frac{\pi}{2}I$,

$$(3.23) \quad \psi(T, x)^{1/2} \ll T^{d/2-\beta d} \gamma_{I, \beta}(T) \alpha_d \left(\Lambda_{T, \frac{2}{\pi}x} \right)^\beta.$$

Note that by Lemma 3.9, part (iii), $\sup_{\omega_{n-1} \in \mathbb{R}} \left(\alpha_d \left(\Lambda_{T, -\frac{2}{\pi}\omega_{n-1}} \right)^\beta \right) \ll T^{d\beta}$. Therefore, by using (3.20) and (3.23), for $\omega \in \frac{\pi}{2}L$ we have

$$\begin{aligned} |\theta_{T, y}(\omega)| &\ll T^{nd-n\beta d} \Gamma_{L_1, \dots, L_{n-1}, \beta}(T) \prod_{i=1}^n \alpha_d \left(\Lambda_{T, \frac{2}{\pi}(\omega_i - \omega_{i-1})} \right)^\beta \\ &\ll T^{nd-(n-1)\beta d} \Gamma_{L_1, \dots, L_{n-1}, \beta}(T) \prod_{i=1}^{n-1} \alpha_d \left(\Lambda_{T, \frac{2}{\pi}(\omega_i - \omega_{i-1})} \right)^\beta. \end{aligned}$$

Note that $L \subset \frac{\pi}{2}L$, hence

$$(3.24) \quad \int_L |\widehat{S}_\epsilon(\omega)| \sup_{y \in \mathbb{R}^{nd}} |\theta_{T, y}(\omega)| d\omega \ll S_\epsilon^*(L) \Gamma_{L_1, \dots, L_{n-1}, \beta}(T) T^{nd-(n-1)\beta d} \prod_{i=1}^{n-1} \int_{L_i - L_{i-1}} \alpha_d(\Lambda_{T, \xi_i})^\beta d\xi_i,$$

where we did the change of variables $\frac{2}{\pi}(\omega_i - \omega_{i-1}) \rightarrow \xi_i$ for $1 \leq i \leq d-1$. (Note, $\frac{2}{\pi}(L_i - L_{i-1}) \subset L_i - L_{i-1}$.) Let $l_{i,0}$ denote the midpoints of the intervals $L_i - L_{i-1}$. Note that $L_i - L_{i-1}$ has length 4. Partition the intervals $L_i - L_{i-1}$ via $l_{i,j} = l_{i,0} - 2 + j \frac{2}{\lambda_{\max}}$ for $j \in [0, 2\lambda_{\max} + 1] \cap \mathbb{Z}$. Note that $d_T u_{\xi_i} = d_{T/\lambda_{\max}} u_{s_{i,j}} d_{\lambda_{\max}} u_{l_{i,j}}$, where $s_{i,j} = (\xi_i - l_{i,j}) \lambda_{\max}^2$. Changing variables from ξ_i to $s_{i,j}$ we get,

$$\begin{aligned} \int_{L_i - L_{i-1}} \alpha_d(\Lambda_{T, \xi_i})^\beta d\xi_i &\ll \sum_{j \in [0, 2\lambda_{\max} + 1] \cap \mathbb{Z}} \int_{[l_{i,j-1}, l_{i,j}]} \alpha_d(d_{T/\lambda_{\max}} u_{s_{i,j}} d_{\lambda_{\max}} u_{l_{i,j}} \Lambda_Q)^\beta ds_{i,j} \\ (3.25) \quad &\ll \max_{j \in [0, 2\lambda_{\max} + 1] \cap \mathbb{Z}} \int_{-2}^2 \alpha_d(d_{T/\lambda_{\max}} u_{s_{i,j}} \Lambda_{Q, l_{i,j}})^\beta ds_{i,j}, \end{aligned}$$

where $\Lambda_{Q, l_{i,j}} = d_{\lambda_{\max}} u_{l_{i,j}} \Lambda_Q$. By using Lemma 3.10 we see that

$$(3.26) \quad \int_{-2}^2 \alpha_d(d_{T/\lambda_{\max}} u_{s_{i,j}} \Lambda_{Q, l_{i,j}})^\beta ds_{i,j} \ll \int_{-\pi}^{\pi} \alpha_d(d_{T/\lambda_{\max}} k_{\theta_{i,j}} \Lambda_{Q, l_{i,j}})^\beta d\theta_{i,j},$$

where $\tan \theta_{i,j} = s_{i,j}$. By Theorem 3.11,

$$(3.27) \quad \int_{-\pi}^{\pi} \alpha_d(d_{T/\lambda_{\max}} k_{\theta_{i,j}} \Lambda_{Q, l_{i,j}})^\beta d\theta_{i,j} \ll \alpha(\Lambda_{Q, l_{i,j}}) T^{\beta d - 2}.$$

Note that Lemma 3.9, part (ii) implies $\alpha(\Lambda_{Q, l_{i,j}}) \ll \alpha_d(\Lambda_{Q, l_{i,j}})$ and Lemma 3.9, part (iii) implies

$$\max_{j \in [0, 2\lambda_{\max} + 1] \cap \mathbb{Z}} \alpha_d(\Lambda_{Q, l_{i,j}}) \ll 1.$$

Hence, using (3.24), (3.25), (3.26) and (3.27) we get the conclusion of the Lemma. \square

We can now prove the bound for E_2 . Recall $T_n = (n-1)T$. Let

$$(3.28) \quad \mathcal{A}_\epsilon(T) = \inf_{\substack{N \in (1, \infty) \\ N- \in (T_n^{-1}, 1)}} \left\{ \log(1/\epsilon)^{n-1} \left(N_-^{d(1/2-\beta)} + \gamma_{[N_-, N], \beta}(T) \right) + \exp \left(-(n-1)c\sqrt{\epsilon N} \right) \right\}.$$

By Lemma 3.9, part (v) for any fixed $\epsilon > 0$ we have $\lim_{T \rightarrow \infty} \mathcal{A}_\epsilon(T) = 0$ provided that Q is irrational.

Proposition 3.13. *For all $0 < \epsilon < 1$, $\tau > 0$ and $T \geq \lambda_{\max}$,*

$$E_2(\tau, \epsilon, T) \ll \|\widehat{\zeta}_\tau\|_1 \mathcal{A}_\epsilon(T) T^{nd-2(n-1)}.$$

Proof. Let $J_{-1} = [0, (2T)^{-1}]$, $J_0 = [(2T)^{-1}, 1]$ and $J_i = [i, i+1]$ for $i \geq 1$. We consider the only the portion of $\mathbb{R}^{n-1} \setminus \mathcal{B}(T)$ lying in the positive cone since bounds for the other cones can be obtained in an identical manner. Note that $\mathbb{R}^{n-1} \setminus \mathcal{B}(T) \subset \mathbb{R}^{n-1} \setminus \left[-(2T)^{-1}, (2T)^{-1} \right]^{n-1}$, therefore the portion of $\mathbb{R}^{n-1} \setminus \mathcal{B}(T)$ lying in the positive cone is contained in

$$\bigcup_{\substack{-1 \leq i_1, \dots, i_{n-1} \\ (i_1, \dots, i_{n-1}) \neq (-1, \dots, -1)}} J_{i_1} \times \dots \times J_{i_{n-1}}.$$

By Lemma 3.12,

$$(3.29) \quad \int_{J_{i_1} \times \dots \times J_{i_{n-1}}} |\widehat{S}_\epsilon(\omega)| \sup_{y \in \mathbb{R}^{nd}} |\theta_{T,y}(\omega)| d\omega \leq S_\epsilon^*(J_{i_1} \times \dots \times J_{i_{n-1}}) \Gamma_{J_{i_1}, \dots, J_{i_{n-1}}, \beta}(T) T^{nd-2(n-1)}.$$

From Lemma 3.3 we have

$$(3.30) \quad S_\epsilon^*(J_{i_1} \times \dots \times J_{i_{n-1}}) \ll \prod_{j=1}^{n-1} \min\{1, |1/i_j|\} \exp\left(-c\sqrt{\epsilon i_j}\right).$$

For $N > 1$,

$$(3.31) \quad \begin{aligned} \sum_{1 \leq i \leq N} \frac{1}{i} \exp\left(-c\sqrt{\epsilon i}\right) &\leq \int_1^N \frac{1}{x} \exp\left(-c\sqrt{\epsilon x}\right) dx = \int_{c\sqrt{\epsilon}}^{c\sqrt{\epsilon N}} \frac{1}{x} \exp(-x) dx \\ &\leq \int_{c\sqrt{\epsilon}}^1 \frac{1}{x} \exp(-x) dx + \int_1^\infty \frac{1}{x} \exp(-x) dx \\ &\ll \log(1/\epsilon). \end{aligned}$$

Hence, (3.30) gives

$$(3.32) \quad \sum_{\substack{-1 \leq i_1, \dots, i_{n-1} \leq N \\ (i_1, \dots, i_{n-1}) \neq (-1, \dots, -1)}} S_\epsilon^*(J_{i_1} \times \dots \times J_{i_{n-1}}) \ll \log(1/\epsilon)^{n-1}.$$

Note that $\{\omega \in \mathbb{R}^{n-1} : |\omega_i - \omega_{i-1}| \leq T_n^{-1}\} \subset \left[-(2T)^{-1}, (2T)^{-1} \right]^{n-1}$ and therefore, for all $-1 \leq i_1, \dots, i_{n-1} \leq N$ with $(i_1, \dots, i_{n-1}) \neq (-1, \dots, -1)$ at least one of the $J_{i_j} - J_{i_{j-1}}$ is contained in $[T_n^{-1}, N]$. From Lemma 3.9, part (iii) it follows that $\gamma_{I, \beta}(T) \ll 1$ for any interval I , moreover, if $I \subset I'$ then $\gamma_{I, \beta}(T) \leq \gamma_{I', \beta}(T)$. Thus, $\Gamma_{J_{i_1}, \dots, J_{i_{n-1}}, \beta}(T) \ll \gamma_{[T_n^{-1}, N], \beta}(T)$. Hence using (3.32),

$$(3.33) \quad \sum_{\substack{-1 \leq i_1, \dots, i_{n-1} \leq N \\ (i_1, \dots, i_{n-1}) \neq (-1, \dots, -1)}} S_\epsilon^*(J_{i_1} \times \dots \times J_{i_{n-1}}) \Gamma_{J_{i_1}, \dots, J_{i_{n-1}}, \beta}(T) \ll \log(1/\epsilon)^{n-1} \gamma_{[T_n^{-1}, N], \beta}(T).$$

Moreover, Lemma 3.9, part (iii) and a similar calculation as (3.31) yields

$$(3.34) \quad \sum_{N < i_1, \dots, i_{n-1}} S_\epsilon^*(J_{i_1} \times \dots \times J_{i_{n-1}}) \Gamma_{J_{i_1}, \dots, J_{i_{n-1}}, \beta}(T) \ll \exp\left(-(n-1)c\sqrt{\epsilon N}\right).$$

Split the interval $[T_n^{-1}, N] = [T_n^{-1}, N_-] \cup [N_-, N]$ where $N_- \in (T_n^{-1}, 1)$. We see that

$$\gamma_{[T_n^{-1}, N], \beta}(T) \leq \gamma_{[T_n^{-1}, N_-], \beta}(T) + \gamma_{[N_-, N], \beta}(T).$$

By Lemma 3.9, part (iv) provided that $\lambda_{\min} \geq n-1$, we get

$$(3.35) \quad \gamma_{[T_n^{-1}, N_-], \beta}(T) \ll N_-^{d(1/2-\beta)}.$$

Using (3.19) combined with the estimates (3.29), (3.33), (3.34) and (3.35) we get that for all $\tau > 0$, $0 < \epsilon < 1$, $T > \lambda_{\max}$, $N_- \in (T_n^{-1}, 1)$ and $N > 1$,

$$E_2(\tau, \epsilon, T) \ll \|\widehat{\zeta}_\tau\|_1 \left(\log(1/\epsilon)^{n-1} \left(N_-^{d(1/2-\beta)} + \gamma_{[N_-, N], \beta}(T) \right) + \exp\left(-(n-1)c\sqrt{\epsilon N}\right) \right) T^{nd-2(n-1)},$$

the claim of the Lemma follows. \square

4. PROOF OF THE MAIN THEOREMS

In this section we combine the results from the previous sections to prove Theorems 1.1 and 1.2. Throughout this section the assertions will hold with the parameter T larger than some constant, which will be called T_0 . However the actual value of T_0 may change from one occurrence to the next. In principle the actual values of T_0 can be determined by analysing the proofs, but we will not do this here.

Lemma 4.1. *For all $\tau \in (0, 1/2)$ and $\epsilon \in (0, 1)$, there exists $T_0 > 0$ such that for all $T > \max(T_0, \tau^{-1})$,*

$$\left| \frac{|\mathbb{Z}^{nd} \cap P_Q^n(I_1, \dots, I_{n-1}) \cap B(T)|}{\text{Vol}(P_Q^n(I_1, \dots, I_{n-1}) \cap B(T))} - 1 \right| \ll \mathcal{E}_{\epsilon, \tau}(T),$$

where $\mathcal{E}_{\epsilon, \tau}(T) = (1/\tau)^{(nd-1)/2} (T^{n-1-d} + \mathcal{A}_\epsilon(T)) + \epsilon + \tau$.

Proof. First note that from Corollary 5.5, there exists $T_0 > 0$ such that for all $T > T_0$, $\text{Vol}(P_Q^n(I_1, \dots, I_{n-1}) \cap B(T)) \gg T^{nd-2(n-1)}$. Therefore

$$(4.1) \quad \left| \frac{|\mathbb{Z}^{nd} \cap P_Q^n(I_1, \dots, I_{n-1}) \cap B(T)|}{\text{Vol}(P_Q^n(I_1, \dots, I_{n-1}) \cap B(T))} - 1 \right| \ll \frac{(|\mathbb{Z}^{nd} \cap P_Q^n(I_1, \dots, I_{n-1}) \cap B(T)| - \text{Vol}(P_Q^n(I_1, \dots, I_{n-1}) \cap B(T)))}{T^{nd-2(n-1)}}.$$

From Lemmas 2.1, 2.2 and Corollaries 5.3 and 5.4 for all $T > T_0$, $\tau \in (0, 1/2)$ and $\epsilon \in (0, 1)$ we get

$$(4.2) \quad \frac{(|\mathbb{Z}^{nd} \cap P_Q^n(I_1, \dots, I_{n-1}) \cap B(T)| - \text{Vol}(P_Q^n(I_1, \dots, I_{n-1}) \cap B(T)))}{T^{nd-2(n-1)}} \ll \frac{\max_{\pm\epsilon, \pm\tau} |R(S_{\pm\epsilon}^Q w_{\pm\tau, T})|}{T^{nd-2(n-1)}} + \epsilon + \tau.$$

From (2.12) and Propositions 3.4, 3.7 and 3.13, we get that for all $\tau \in (0, 1/2)$, $\epsilon \in (0, 1)$ and $T \geq \max(\tau^{-1}, T_0)$.

$$(4.3) \quad \frac{\max_{\pm\epsilon, \pm\tau} |R(S_{\pm\epsilon}^Q w_{\pm\tau, T})|}{T^{nd-2(n-1)}} \ll \|\widehat{\zeta}_\tau\|_1 T^{2-d} + T^{n-1-d} (\|\widehat{\zeta}_\tau\|_1 + (1/\tau)^{(nd-1)/2}) + \|\widehat{\zeta}_\tau\|_1 \mathcal{A}_\epsilon(T).$$

Note that $2-d \leq n-1-d$ for $n \geq 3$. Finally, using Corollary 5.7 to bound $\|\widehat{\zeta}_\tau\|_1 \ll (1/\tau)^{(nd-1)/2}$ we get the conclusion of the Lemma from (4.1), (4.2) and (4.3). \square

The proof of Theorem 1.1 follows immediately from Lemma 4.1.

Proof of Theorem 1.1. Note that for $n \leq d$ and all $\tau \in (0, 1/2)$ and $\epsilon \in (0, 1)$,

$$\lim_{T \rightarrow \infty} \left((1/\tau)^{(nd-1)/2} (T^{n-1-d} + \mathcal{A}_\epsilon(T)) + \epsilon + \tau \right) = (1/\tau)^{(nd-1)/2} \lim_{T \rightarrow \infty} \mathcal{A}_\epsilon(T) + \epsilon + \tau.$$

By Lemma 3.9, part (v) for all $\epsilon > 0$, $\lim_{T \rightarrow \infty} \mathcal{A}_\epsilon(T) = 0$, when Q is irrational. Hence, for all $\tau \in (0, 1/2)$ and $\epsilon \in (0, 1)$, $\lim_{T \rightarrow \infty} \mathcal{E}_{\epsilon, \tau}(T) = \epsilon + \tau$ and this is the claim of Theorem 1.1. \square

Note that in order to prove Theorem 1.1 we did not need to use bounds for $\|\widehat{\zeta}_\tau\|_1$ from subsection 5.2. It was only necessary to know that for any $\tau > 0$ the quantities involving τ remain finite. In order to prove Theorem 1.2 we will use the bounds from subsection 5.2 together with bounds from Section 3 and results from [GM13] (see Lemma 3.9, part (vi)) that give an explicit rate for the convergence of $\lim_{T \rightarrow \infty} \gamma_{[N_-, N_+], \beta}(T)$ when Q is of Diophantine type (κ, A) .

Proof of Theorem 1.2. Recall the definition of $\mathcal{A}_\epsilon(T)$. Take $N_+ = \frac{\log^2(1/\epsilon)}{(n-1)^2 \epsilon c^2}$. Then $\exp(-(n-1)c\sqrt{\epsilon N_+}) = \epsilon$. Then for all $\tau \in (0, 1/2)$, $\epsilon \in (0, 1)$ and $T \geq 1$

$$\mathcal{E}_{\epsilon, \tau}(T) \leq \inf_{N_- \in (T_n^{-1}, 1)} \left\{ (1/\tau)^{(nd-1)/2} \log^{n-1}(1/\epsilon) \left(N_-^{d(1/2-\beta)} + \gamma_{[N_-, N_+], \beta}(T) + T^{n-1-d} \right) + \epsilon + \tau \right\}.$$

Let $\sigma > 0$, set $\tau = \epsilon = N_-^\sigma$, $(nd-1)/2 = q$, $1/2 - \beta = \beta'$, then for all $T \geq 1$,

$$\mathcal{E}_{\epsilon, \tau}(T) \leq \inf_{N_- \in (T_n^{-1}, 1)} \left\{ |\log(N_-)|^{n-1} \left(N_-^{\beta'd-\sigma q} + N_-^{-\sigma q} \gamma_{[N_-, N_+], \beta}(T) + N_-^\sigma + N_-^{-\sigma q} T^{n-1-d} \right) \right\}.$$

By Lemma 3.9, part (vi), if Q is of type (κ, A) , then $\gamma_{[N_-, N_+], \beta}(T) \ll (\max\{N_-^{-1-\kappa}, N_+^\kappa\} T^{-2(1-\kappa)})^{\beta'}$ for all $T \geq 1$. Note that

$$\max\{N_-^{-1-\kappa}, N_+^\kappa\} \ll \max\left\{ \frac{1}{N_-^{1+\kappa}}, \frac{|\log(N_-)|^{2\kappa}}{N_-^{\kappa\sigma}} \right\} \ll \frac{1}{N_-^{1+\kappa}},$$

provided that $\sigma < 1$. Let $2(1 - \kappa) = \kappa'$, then for all $T \geq 1$,

$$(4.4) \quad \mathcal{E}_{\epsilon, \tau, T} \ll \inf_{N_- \in (T_n^{-1}, 1)} \left\{ |\log(N_-)|^{n-1} \mathcal{N}_{\sigma, \kappa, \beta}(N_-, T) \right\},$$

where $\mathcal{N}_{\sigma, \kappa, \beta}(N_-, T) = N_-^{\beta'd - \sigma q} + N_-^{-\sigma q - (1+\kappa)\beta'} T^{-\kappa'\beta'} + N_-^\sigma + N_-^{-\sigma q} T^{n-1-d}$. Let $N_- = T^{\frac{-\kappa'\beta'}{\sigma + \sigma q + (1+\kappa)\beta'}}$ for $\sigma \geq \frac{2\beta'}{nd+1}$, we have $0 < \frac{\kappa'\beta'}{\sigma + \sigma q + (1+\kappa)\beta'} \leq 1$. Thus, there exists $T_0 > 0$ such that for $T > T_0$ we have $N_- \in (T_n^{-1}, 1)$ and

$$(4.5) \quad \mathcal{N}_{\sigma, \kappa, \beta}(N_-, T) = T^{\frac{-\kappa'\beta'(\beta'd - \sigma q)}{\sigma + \sigma q + (1+\kappa)\beta'}} + 2T^{\frac{-\kappa'\beta'\sigma}{\sigma + \sigma q + (1+\kappa)\beta'}} + T^{\frac{\sigma q \kappa' \beta'}{\sigma + \sigma q + (1+\kappa)\beta'} + n-1-d}.$$

Note that for $n \leq d$,

$$\frac{\sigma q \kappa' \beta'}{\sigma + \sigma q + (1+\kappa)\beta'} + n - 1 - d \leq \frac{\sigma q \kappa' \beta'}{\sigma + \sigma q + (1+\kappa)\beta'} - 1 = \frac{\sigma q \kappa' \beta' - \sigma - \sigma q - (1+\kappa)\beta'}{\sigma + \sigma q + (1+\kappa)\beta'}$$

and since

$$\sigma q \kappa' \beta' - \sigma - \sigma q - (1+\kappa)\beta' + \kappa'\beta'\sigma = \sigma(1+q)(\kappa'\beta' - 1) - (1+\kappa)\beta' < 0,$$

we get

$$\frac{\sigma q \kappa' \beta'}{\sigma + \sigma q + (1+\kappa)\beta'} + n - 1 - d < \frac{-\kappa'\beta'\sigma}{\sigma + \sigma q + (1+\kappa)\beta'}.$$

Thus, from (4.5) we get

$$(4.6) \quad \mathcal{N}_{\sigma, \kappa, \beta}(N_-, T) \ll T^{\frac{-\kappa'\beta'(\beta'd - \sigma q)}{\sigma + \sigma q + (1+\kappa)\beta'}} + 2T^{\frac{-\kappa'\beta'\sigma}{\sigma + \sigma q + (1+\kappa)\beta'}}.$$

Let $\sigma = \frac{\beta'd}{1+q}$, then $\frac{2\beta'}{nd+1} \leq \sigma < 1$, and (4.6) becomes

$$\mathcal{N}_{\sigma, \kappa, \beta}(N_-, T) \ll T^{-\frac{4d\beta'(1-\kappa)}{(1+nd)(d+1+\kappa)}}.$$

Thus, using this and (4.4) we get that there exists $T_0 > 0$ such that for all $T > T_0$,

$$\mathcal{E}_{\epsilon, \tau}(T) \ll \log^{n-1}(T) T^{-\delta},$$

where

$$\delta = \sup_{\beta' \in (0, 1/2 - 2/d)} \frac{4d\beta'(1-\kappa)}{(1+nd)(d+1+\kappa)} > \frac{2(d-4)(1-\kappa)}{(1+nd)(d+1+\kappa)} - \varsigma$$

for any $\varsigma > 0$. Finally, we can apply Lemma 4.1, to get that for all $T > \max(T_0, \tau^{-1})$,

$$\left| \frac{|\mathbb{Z}^{nd} \cap P_Q^n(I_1, \dots, I_{n-1}) \cap B(T)|}{\text{Vol}(P_Q^n(I_1, \dots, I_{n-1}) \cap B(T))} - 1 \right| \ll \log^{n-1}(T) T^{-\delta}.$$

We chose $\tau = N_-^\sigma$ and then $N_- = T^{\frac{-\kappa'\beta'}{\sigma + \sigma q + (1+\kappa)\beta'}}$, therefore $\tau^{-1} = T^{\frac{\sigma \kappa' \beta'}{\sigma + \sigma q + (1+\kappa)\beta'}}$. Because $\frac{\sigma \kappa' \beta'}{\sigma + \sigma q + (1+\kappa)\beta'} < 1$, the condition that $T > \tau^{-1}$ is automatically satisfied for the choices that were made. This completes the proof the Theorem. \square

5. VOLUME AND NORM ESTIMATES

5.1. Volume estimates. In order to bound the smoothing errors coming from Lemmas 2.1 and 2.2 we need to estimate the volumes of certain regions of \mathbb{R}^{nd} . Similar computations were done in [Mül08] (Lemma 5) for positive definite forms and in [EMM98] (Lemma 3.8) and [GM13] (Lemma 7.1) for a single quadratic form. For $m \in \mathbb{N}$, let S_m be the unit sphere in \mathbb{R}^m . For $g \in GL_d(\mathbb{R})$, and $v \in \mathbb{R}^{nd}$, let $gv = (gv_1, \dots, gv_n)$. Let f_1 and f_2 be compactly supported functions on \mathbb{R} and \mathbb{R}^{n-1} respectively. Let

$$\Theta(f_1, f_2, T) = \int_{\mathbb{R}^{nd}} f_2(Q(v_1) - Q(v_2), \dots, Q(v_{n-1}) - Q(v_n)) f_1(T^{-1}\|v\|) dv.$$

There exists $g_0 \in GL_d(\mathbb{R})$ such that $Q(v_i) = Q_0(g_0 v_i)$, where Q_0 is equal to a diagonal form with coefficients equal to ± 1 . Suppose that the signature of Q_0 is (p, q) . Since $d \geq 5$, without loss of generality we may suppose that $q \geq 2$. By making the change of variables $y_i = g_0 v_i$, we get

$$\Theta(f_1, f_2, T) = d_Q \int_{\mathbb{R}^{nd}} f_2(Q_0(y_1) - Q_0(y_2), \dots, Q_0(y_{n-1}) - Q_0(y_n)) f_1(T^{-1}\|g_0^{-1}y\|) dy,$$

where $d_Q = 1/\det(g_0)^n$. Suppose $p \geq 1$ (i.e. Q is indefinite). We will work in polar coordinates. We can write $T^{-1}y = (\rho_1 \eta_1, \dots, \rho_{2n} \eta_{2n})$ where $\rho = (\rho_1, \dots, \rho_{2n}) \in [0, \infty)^{2n}$ and $\eta = (\eta_1, \dots, \eta_{2n}) \in (S_p \times S_q)^n$. It follows

that $Q_0(y_i) = T^2(\rho_{2i-1}^2 - \rho_{2i}^2)$. Let $f_g(v) = f_1(\|g^{-1}v\|)$ and $\bar{\rho}_p = (\prod_{i=1}^n \rho_{2i-1})^{p-1}$, $\bar{\rho}_q = (\prod_{i=1}^n \rho_{2i})^{q-1}$ and $Q_i(\rho) = \rho_{2i-1}^2 - \rho_{2i}^2 - \rho_{2i+1}^2 + \rho_{2i+2}^2$. Then

$$(5.1) \quad \Theta(f_1, f_2, T) = d_Q T^{nd} \int_{[0, \infty)^{2n}} \bar{\rho}_p \bar{\rho}_q f_2(T^2 Q_1(\rho), \dots, T^2 Q_{n-1}(\rho)) \Psi_{f_1}(\rho) d\rho,$$

where

$$(5.2) \quad \Psi_{f_1}(\rho) = \int_{(S_p \times S_q)^n} f_{g_0}(\rho_1 \eta_1, \dots, \rho_{2n} \eta_{2n}) d\eta.$$

The following Lemma will be used to obtain the required bounds for the smoothing errors.

Lemma 5.1. *Let f_1 be a continuous, compactly supported function on \mathbb{R} and V be a bounded Borel measurable subset of \mathbb{R}^{n-1} . Then, there exists a positive constant C_{f_1} , such that*

$$\lim_{T \rightarrow \infty} \frac{1}{T^{nd-2(n-1)}} \Theta(f_1, \mathbb{1}_V, T) = C_{f_1} \text{Vol}(V).$$

Proof. Change variables in the equations (5.1) and (5.2) by letting $u = F(\rho)$, where F is defined by

$$(5.3) \quad u_i = \begin{cases} \rho_i & \text{if } i \text{ is odd or } i = 2n \\ T^2(\rho_{i-1}^2 - \rho_i^2) & \text{if } i \text{ is even and } i \neq 2n. \end{cases}$$

Note that the Jacobian of F is given by $2^{n-1} T^{2(n-1)} \prod_{i=1}^{n-1} |\rho_{2i}|$. Therefore

$$|\bar{\rho}_p \bar{\rho}_q| d\rho = 2^{1-n} T^{2(1-n)} \left(\prod_{i=1}^n |\rho_{2i-1}| \right)^{p-1} \left(\prod_{i=1}^n |\rho_{2i}| \right)^{q-2} |\rho_{2n}| du.$$

Moreover, we can write

$$\rho_i = \begin{cases} u_i & \text{if } i \text{ is odd or } i = 2n \\ \sqrt{u_{i-1}^2 - u_i/T^2} & \text{if } i \text{ is even and } i \neq 2n. \end{cases}$$

Therefore

$$(5.4) \quad \Theta(f_1, f_2, T) = 2^{1-n} d_Q T^{nd-2(n-1)} \int_{F([0, \infty)^{2n})} f_2(u_2 - u_4, \dots, u_{2n-2} - u_{2n}) \bar{\Psi}_{f_1}(u, T) du,$$

where

$$(5.5) \quad \bar{\Psi}_{f_1}(u, T) = J(u, T) \int_{(S_p \times S_q)^n} f_1 \left(\left\| g_0^{-1} \left(u_1 \eta_1, \sqrt{u_1^2 - u_2/T^2} \eta_2, \dots, u_{2n} \eta_{2n} \right) \right\| \right) d\eta$$

and

$$J(u, T) = \left(\prod_{i=1}^n |u_{2i-1}| \right)^{p-1} \left(\prod_{i=1}^n |u_{2i-1}^2 - u_{2i}/T^2| \right)^{\frac{q-2}{2}} |u_{2n}|.$$

Note that $F([0, \infty)^{2n}) = ([0, \infty) \times \mathbb{R})^{n-1} \times [0, \infty) \times [0, \infty)$. Since f_1 is continuous with compact support, f_1 can be bounded by an integrable function and hence by the Dominated Convergence Theorem

$$\lim_{T \rightarrow \infty} \bar{\Psi}_{f_1}(u, T) = \lim_{T \rightarrow \infty} J(u, T) \int_{(S_p \times S_q)^n} \lim_{T \rightarrow \infty} f_1 \left(\left\| g_0^{-1} \left(u_1 \eta_1, \sqrt{u_1^2 - u_2/T^2} \eta_2, \dots, u_{2n} \eta_{2n} \right) \right\| \right) d\eta.$$

Since f_1 is continuous we get

$$(5.6) \quad \lim_{T \rightarrow \infty} \bar{\Psi}_{f_1}(u, T) = \left(\prod_{i=1}^n |u_{2i-1}| \right)^{p+q-3} |u_{2n}| \int_{(S_p \times S_q)^n} f_1(\|g_0^{-1}(u_1 \eta_1, u_1 \eta_2, \dots, u_{2n} \eta_{2n})\|) d\eta.$$

Hence, we see that $\lim_{T \rightarrow \infty} \bar{\Psi}_{f_1}(u, T)$ depends only on u_i if i is odd or $i = 2n$. Let $\bar{\bar{\Psi}}_{f_1}(u_1, \dots, u_{2n-1}, u_{2n}) = \lim_{T \rightarrow \infty} \bar{\Psi}_{f_1}(u, T)$. Because f_1 has compact support, it follows that the support of $\bar{\bar{\Psi}}_{f_1}(u_1, \dots, u_{2n-1}, u_{2n})$ is also compact. Let $g_1 \in SL_{2n}(\mathbb{R})$ be such that $u' = g_1 u$ where

$$u'_i = \begin{cases} u_i & \text{if } i \text{ is odd or } i = 2n \\ u_i - u_{i+2} & \text{if } i \text{ is even and } i \neq 2n. \end{cases}$$

Note that $du = du'$ and $g_1 F([0, \infty)^{2n}) = F([0, \infty)^{2n})$. Let $du'_e = \prod_{i=1}^{n-1} du_{2i}$, $du'_o = \prod_{i=1}^n du_{2i-1} du_{2n}$. The fact that f_2 has compact support and thus can be bounded by an integrable function, means that the Dominated Convergence Theorem, together with (5.4) and (5.6) yields

$$(5.7) \quad \lim_{T \rightarrow \infty} \frac{1}{T^{nd-2(n-1)}} \Theta(f_1, f_2, T) = 2^{1-n} d_Q \int_{\mathbb{R}^{n-1}} f_2(u'_2, \dots, u'_{2n-2}) du'_e \int_{[0, \infty)^{n+1}} \overline{\Psi}_{f_1}(u'_1, \dots, u'_{2n-1}, u'_{2n}) du'_o.$$

Therefore, by setting $f_2 = \mathbb{1}_V$ we get

$$\lim_{T \rightarrow \infty} \frac{1}{T^{nd-2(n-1)}} \Theta(f_1, \mathbb{1}_V, T) = C_{f_1} \text{Vol}(V),$$

where

$$(5.8) \quad C_{f_1} = 2^{1-n} d_Q \int_{[0, \infty)^{n+1}} \overline{\Psi}_{f_1}(u'_1, \dots, u'_{2n-1}, u'_{2n}) du'_o$$

is a positive constant, as required. \square

Remark 5.2. The case when $p = 0$ (i.e. Q is negative definite) can be dealt with in the same way up to minor modifications of coordinates involved. Specifically, we write $T^{-1}y = (\rho_1 \eta_1, \dots, \rho_n \eta_n)$, where $\rho = (\rho_1, \dots, \rho_n) \in [0, \infty)^2$ and $\eta = (\eta_1, \dots, \eta_n) \in S_{d-1}^n$. It follows that $Q_0(y_i) = T^2 \rho_i^2$. The change of variables (5.3) also needs to be replaced by

$$u_i = \begin{cases} \rho_i^2 - \rho_{i+1}^2 & \text{if } i < n \\ \rho_i & \text{if } i = n. \end{cases}$$

The Jacobian is then $2^{n-1} \prod_{i=1}^{n-1} |\rho_i|$, and it is straightforward to check that the rest of the proof remains intact for this situation. See also Lemma 5 of [Mül08].

Corollary 5.3. *There exists $T_0 > 0$ such that, for all $\tau \in (0, 1/2)$ and $T > T_0$,*

$$\int_{\mathbb{R}^{nd}} (\mathbb{1}_{B(1+2\tau)} - \mathbb{1}_{B(1-2\tau)}) d\nu_T \ll \tau T^{nd-2(n-1)}.$$

Proof. Note that $\int_{\mathbb{R}^{nd}} (\mathbb{1}_{B(1+2\tau)} - \mathbb{1}_{B(1-2\tau)}) d\nu_T = \Theta(\mathbb{1}_{[1-2\tau, 1+2\tau]}, \mathbb{1}_{I_1 \times \dots \times I_{n-1}}, T)$. For all $\tau > 0$, there exists a continuous function f_τ , such that $f_\tau(x) = 1$ for all $x \in [1-2\tau, 1+2\tau]$ and $f_\tau(x) = 0$ for all $x \notin [1-3\tau, 1+3\tau]$. We have

$$\Theta(\mathbb{1}_{[1-2\tau, 1+2\tau]}, \mathbb{1}_{I_1 \times \dots \times I_{n-1}}, T) \leq \Theta(f_\tau, \mathbb{1}_{I_1 \times \dots \times I_{n-1}}, T).$$

Therefore, in view of Lemma 5.1 we must show that $C_{f_\tau} \ll \tau$. Using (5.6) and (5.8) we have

$$(5.9) \quad C_{f_\tau} \ll \int_{[0, \infty)^{n+1}} \left(\left(\prod_{i=1}^n |u_{2i-1}| \right)^{p+q-3} |u_{2n}| \int_{(S_p \times S_q)^n} \mathbb{1}_{[1-3\tau, 1+3\tau]}(\|g_0^{-1}(u_1 \eta_1, u_1 \eta_2, \dots, u_{2n} \eta_{2n})\|) d\eta \right) du_o.$$

If u is such that $\mathbb{1}_{[1-3\tau, 1+3\tau]}(\|g_0^{-1}(u_1 \eta_1, u_1 \eta_2, \dots, u_{2n} \eta_{2n})\|)$ is non zero then u is in a bounded subset of \mathbb{R}^{n+1} . Hence

$$(5.10) \quad C_{f_\tau} \ll \int_{[0, \infty)^{n+1}} \left(\int_{(S_p \times S_q)^n} \mathbb{1}_{[1-3\tau, 1+3\tau]}(\|g_0^{-1}(u_1 \eta_1, u_1 \eta_2, \dots, u_{2n} \eta_{2n})\|) d\eta \right) du_o.$$

Next we change variables by letting $u_i = r \rho_i$, where $r \geq 0$ and $\rho_o = (\rho_1, \rho_3, \dots, \rho_{2n}) \in S_{n+1}$. We get

$$(5.11) \quad C_{f_\tau} \ll \int_{[0, \infty)} r^n \left(\int_{S_{n+1}} \left(\int_{(S_p \times S_q)^n} \mathbb{1}_{[1-3\tau, 1+3\tau]}(r \|g_0^{-1}(\rho_1 \eta_1, \rho_1 \eta_2, \dots, \rho_{2n} \eta_{2n})\|) d\eta \right) d\rho_o \right) dr.$$

Let $N(\eta, \rho_o) = \|g_0^{-1}(\rho_1 \eta_1, \rho_1 \eta_2, \dots, \rho_{2n} \eta_{2n})\|$. Using Fubini's Theorem to change the order of integration in (5.11) we get

$$\begin{aligned} C_{f_\tau} &\ll \int_{S_{n+1}} \int_{(S_p \times S_q)^n} \int_{(1-3\tau)/N(\eta, \rho_o)}^{(1+3\tau)/N(\eta, \rho_o)} r^n dr d\eta d\rho_o \\ &\ll \tau \int_{S_{n+1}} \int_{(S_p \times S_q)^n} N(\eta, \rho_o)^{-(n+1)} d\eta d\rho_o. \end{aligned}$$

By compactness $\int_{S_{n+1}} \int_{(S_p \times S_q)^n} N(\eta, \rho_o)^{-(n+1)} d\eta d\rho_o \ll 1$, from which the claim of the Lemma follows. \square

Corollary 5.4. *There exists $T_0 > 0$ such that, for all $\tau \in (0, 1)$, $\epsilon \in (0, 1)$ and $T > T_0$,*

$$\int_{\mathbb{R}^{n-1}} \left(\mathbb{1}_{I_1^{2\epsilon} \times \dots \times I_{n-1}^{2\epsilon}} - \mathbb{1}_{I_1^{-2\epsilon} \times \dots \times I_{n-1}^{-2\epsilon}} \right) d\nu_{\tau, T} \ll \epsilon T^{nd-2(n-1)}.$$

Proof. Note that for all $\tau \in (0, 1)$, w_τ is a continuous function with compact support on \mathbb{R}^{nd} , therefore there exists a continuous function, f with compact support on \mathbb{R} , such that $w_\tau(v) \leq f(\|v\|)$ for all $v \in \mathbb{R}^{nd}$. Then $\int_{\mathbb{R}^{n-1}} \left(\mathbb{1}_{I_1^{2\epsilon} \times \dots \times I_{n-1}^{2\epsilon}} - \mathbb{1}_{I_1^{-2\epsilon} \times \dots \times I_{n-1}^{-2\epsilon}} \right) d\nu_{\tau, T} \leq \Theta \left(f, \mathbb{1}_{I_1^{2\epsilon} \times \dots \times I_{n-1}^{2\epsilon}} - \mathbb{1}_{I_1^{-2\epsilon} \times \dots \times I_{n-1}^{-2\epsilon}}, T \right)$. For all $\epsilon \in (0, 1)$ we have $\text{Vol} \left(\text{supp} \left(\mathbb{1}_{I_1^{2\epsilon} \times \dots \times I_{n-1}^{2\epsilon}} - \mathbb{1}_{I_1^{-2\epsilon} \times \dots \times I_{n-1}^{-2\epsilon}} \right) \right) \ll \epsilon$, therefore the corollary follows by applying Lemma 5.1. \square

Corollary 5.5. *There exists a positive constant $C_{Q,n}$, depending only on Q and n , such that*

$$\lim_{T \rightarrow \infty} \frac{1}{T^{nd-2(n-1)}} \text{Vol} \left(P_Q^n(I_1, \dots, I_{n-1}) \cap B(T) \right) = C_{Q,n} \prod_{i=1}^{n-1} |I_i|.$$

Proof. Note that $\text{Vol} \left(P_Q^n(I_1, \dots, I_{n-1}) \cap B(T) \right) = \Theta \left(\mathbb{1}_{[0,1]}, \mathbb{1}_{I_1 \times \dots \times I_{n-1}}, T \right)$. The conclusion follows from Lemma 5.1 by the standard trick of approximating $\mathbb{1}_{[0,1]}$ from above and below by continuous functions. \square

5.2. Norm estimates. In this subsection we prove estimates for $\|\widehat{\zeta}_\tau\|_1$ and the related quantity that appeared in the proof of Proposition 3.7. The estimates follow from standard results about Bessel functions. Note that for any fixed $\tau > 0$ the fact that $\widehat{\zeta}_\tau \in \mathcal{S}(\mathbb{R}^{nd})$ implies that $\|\widehat{\zeta}_\tau\|_1 < \infty$. In order to prove Theorem 1.1, this is the only information regarding $\|\widehat{\zeta}_\tau\|_1$ that is required. However, in order to prove Theorem 1.2, an explicit bound for $\|\widehat{\zeta}_\tau\|_1$ is required. The required bound will follow from an estimate of $\|\widehat{w}_\tau\|_1$.

Lemma 5.6. *For all $0 < \tau < 1$, $\|\widehat{w}_\tau\|_1 \ll \tau^{(1-nd)/2}$.*

Proof. From the definitions of w_τ and k_τ^{nd} it follows that

$$(5.12) \quad \|\widehat{w}_\tau\|_1 = \int_{\mathbb{R}^{nd}} \left| \widehat{\mathbb{1}}_{[0,1]^\tau}(\|v\|) \widehat{k_\tau^{nd}}(v) \right| dv \leq \int_{\mathbb{R}^{nd}} \left| \widehat{\mathbb{1}}_{[0,1]^\tau}(\|v\|) \right| \exp \left(-c\sqrt{\tau\|v\|} \right) dv.$$

Let \mathcal{J}_μ denote the Bessel function of order μ . See Appendix B page 425 of [Gra08]. Using the results from B3 and B5 of [Gra08] it follows that

$$(5.13) \quad \left| \widehat{\mathbb{1}}_{[0,1]^\tau}(\|v\|) \right| = \left| \frac{\mathcal{J}_{nd/2}(2\pi(1+\tau)\|v\|)(1+\tau)^{nd/2-2}}{\|v\|^{nd/2}} \right| \ll \left| \frac{\mathcal{J}_{nd/2}(2\pi(1+\tau)\|v\|)}{\|v\|^{nd/2}} \right|.$$

By B8 of [Gra08] we have $\mathcal{J}_{nd/2}(2\pi(1+\tau)\|v\|) \ll (2\pi(1+\tau)\|v\|)^{-1/2} \ll \|v\|^{-1/2}$ for $\|v\| \geq (2\pi(1+\tau))^{-1}$. Let $r = (2\pi(1+\tau))^{-1}$. Therefore, from (5.12) and (5.13) we get

$$(5.14) \quad \begin{aligned} \|\widehat{w}_\tau\|_1 &\ll \int_{\mathbb{R}^{nd}} \left| \frac{\mathcal{J}_{nd/2}(2\pi(1+\tau)\|v\|)}{\|v\|^{nd/2}} \right| \exp \left(-c\sqrt{\tau\|v\|} \right) dv \\ &\ll \int_{B(r)} \frac{\mathcal{J}_{nd/2}(2\pi(1+\tau)\|v\|)}{\|v\|^{nd/2}} \exp \left(-c\sqrt{\tau\|v\|} \right) dv + \int_{\mathbb{R}^{nd} \setminus B(r)} \frac{\exp \left(-c\sqrt{\tau\|v\|} \right)}{\|v\|^{(nd+1)/2}} dv. \end{aligned}$$

Moreover, by B6 of [Gra08] we get that $\mathcal{J}_{nd/2}(2\pi(1+\tau)\|v\|) / \|v\|^{nd/2}$ is bounded for all $0 < \|v\| \leq r$. Also, from the definition in B1 of [Gra08], it follows that when $\|v\| = 0$, this quantity is also bounded. Therefore

$$(5.15) \quad \int_{B(r)} \frac{\mathcal{J}_{nd/2}(2\pi(1+\tau)\|v\|)}{\|v\|^{nd/2}} \exp \left(-c\sqrt{\tau\|v\|} \right) dv \ll 1.$$

Since $r \leq 1$, by changing to polar coordinates we get

$$\begin{aligned} \int_{\mathbb{R}^{nd} \setminus B(r)} \frac{\exp \left(-c\sqrt{\tau\|v\|} \right)}{\|v\|^{(nd+1)/2}} dv &\ll \int_1^\infty x^{(nd-3)/2} \exp \left(-c\sqrt{\tau x} \right) dx \\ &\ll \tau^{(1-nd)/2} \int_{c\tau^{1/2}}^\infty x^{nd-2} \exp(-x) dx \\ &\ll \tau^{(1-nd)/2}. \end{aligned}$$

Combining this with (5.14) and (5.15) we get the conclusion of the Lemma. \square

Corollary 5.7. *For all $0 < \tau < 1$, $\|\widehat{\zeta}_\tau\|_1 \ll \tau^{(1-nd)/2}$.*

Proof. Let φ be a smooth function on \mathbb{R}^{nd} such that $\varphi(v) = 1$ for all v in the support of w_τ and $\varphi(v) = 0$ if $\|v\| > 3$. It follows that $\zeta_\tau(v) = w_\tau(v) \exp(Q_{++}(v)) \varphi(v)$. Let $\chi(v) = \exp(Q_{++}(v)) \varphi(v)$. Note that because $\varphi \in C_0^\infty(\mathbb{R}^{nd})$ and $\exp(Q_{++}(v))$ is bounded on the support of φ we have that $\chi \in \mathcal{S}(\mathbb{R}^{nd})$. Therefore $\widehat{\chi} \in \mathcal{S}(\mathbb{R}^{nd})$ and hence

$$(5.16) \quad \|\widehat{\chi}\|_1 < \infty.$$

It is a standard fact (for example see Proposition 2.2.11 (12) of [Gra08]) that

$$\int_{\mathbb{R}^{nd}} |\widehat{\zeta}_\tau(y)| dy \leq \int_{\mathbb{R}^{nd}} \int_{\mathbb{R}^{nd}} |\widehat{w}_\tau(y-v)| |\widehat{\chi}(v)| dv dy = \|\widehat{\chi}\|_1 \|\widehat{w}_\tau\|_1.$$

Thus, the conclusion of the Corollary follows from Lemma 5.6 and (5.16). \square

The following Lemma was used in the proof of Proposition 3.7. Again we remark that knowledge of the exact dependence on τ is not necessary if one only wants to prove Theorem 1.1.

Lemma 5.8. *For all $0 < \tau < 1$ and $T \geq \tau^{-1}$,*

$$\int_{\mathbb{R}^{nd} \setminus B_\infty(T/2)} |\widehat{\zeta}_\tau(y)| dy \ll \frac{1}{T^d} \tau^{(1-nd)/2}.$$

Proof. Again, using standard results

$$(5.17) \quad \int_{\mathbb{R}^{nd} \setminus B_\infty(T/2)} |\widehat{\zeta}_\tau(y)| dy = \int_{\mathbb{R}^{nd} \setminus B_\infty(T/2)} \left(\int_{B_\infty(T/4)} |\widehat{w}_\tau(y-v)| |\widehat{\chi}(v)| dv + \int_{\mathbb{R}^{nd} \setminus B_\infty(T/4)} |\widehat{w}_\tau(y-v)| |\widehat{\chi}(v)| dv \right) dy.$$

It is clear that

$$(5.18) \quad \int_{\mathbb{R}^{nd} \setminus B_\infty(T/2)} \int_{\mathbb{R}^{nd} \setminus B_\infty(T/4)} |\widehat{w}_\tau(y-v)| |\widehat{\chi}(v)| dv dy \leq \|\widehat{w}_\tau\|_1 \int_{\mathbb{R}^{nd} \setminus B_\infty(T/4)} |\widehat{\chi}(v)| dv$$

and by changing variables $y - v = y'$, we see that if $v \in B_\infty(T/4)$ and $y \in B_\infty(T/2)$, then $y' \in \mathbb{R}^{nd} \setminus B_\infty(T/4)$ and hence

$$(5.19) \quad \int_{\mathbb{R}^{nd} \setminus B_\infty(T/2)} \int_{B_\infty(T/4)} |\widehat{w}_\tau(y-v)| |\widehat{\chi}(v)| dv dy \leq \|\widehat{\chi}\|_1 \int_{\mathbb{R}^{nd} \setminus B_\infty(T/4)} |\widehat{w}_\tau(y)| dy.$$

Therefore, by combining (5.17), (5.18) and (5.19) we get

$$(5.20) \quad \int_{\mathbb{R}^{nd} \setminus B_\infty(T/2)} |\widehat{\zeta}_\tau(y)| dy \leq \|\widehat{\chi}\|_1 \int_{\mathbb{R}^{nd} \setminus B_\infty(T/4)} |\widehat{w}_\tau(y)| dy + \|\widehat{w}_\tau\|_1 \int_{\mathbb{R}^{nd} \setminus B_\infty(T/4)} |\widehat{\chi}(v)| dv.$$

Moreover, because $\widehat{\chi} \in \mathcal{S}(\mathbb{R}^{nd})$, for all $k \in \mathbb{N}$ there exists a constant c_k such that $|\widehat{\chi}(v)| \ll c_k (1 + \|v\|^2)^{-k}$. By taking k large enough (For instance, $k = \frac{(n-1)d}{2} - 1$.) it follows that

$$(5.21) \quad \int_{\mathbb{R}^{nd} \setminus B_\infty(T/4)} |\widehat{\chi}(v)| dv \ll \frac{1}{T^d}.$$

We can use the method of Lemma 5.6 to get

$$(5.22) \quad \begin{aligned} \int_{\mathbb{R}^{nd} \setminus B_\infty(T/4)} |\widehat{w}_\tau(y)| dy &\ll \int_{\mathbb{R}^{nd} \setminus B_\infty(T/4)} \left| \frac{\mathcal{J}_{nd/2}(2\pi(1+\tau)\|v\|)}{\|v\|^{nd/2}} \right| \exp(-c\sqrt{\tau}\|v\|) dv \\ &\ll \int_{T/4}^\infty r^{(nd-3)/2} \exp(-c\sqrt{\tau}r) dr \\ &\ll \tau^{(1-nd)/2} \int_{Tc\tau^{1/2}/4}^\infty x^{nd-2} \exp(-x) dx. \end{aligned}$$

For all $k \in \mathbb{N}$ there exists a constant c_k such that for $x \geq 0$, $|x^{nd-2} \exp(-x)| \leq c_k (1+x)^{-k}$. It follows that

$$(5.23) \quad \int_{Tc\tau^{1/2}/4}^\infty x^{nd-2} \exp(-x) dx \ll_k \left(\frac{1}{\tau}\right)^{k/2} \frac{1}{T^k}$$

for any $k \in \mathbb{N}$. Therefore, if $k > d$ and $T \geq \tau^{\frac{-k}{2k-2d}}$, using (5.22) and (5.23) we see

$$(5.24) \quad \int_{\mathbb{R}^{nd} \setminus B_\infty(T/4)} |\widehat{w}_\tau(y)| dy \ll_k \frac{1}{T^d}.$$

The conclusion of the Lemma follows by choosing $k = 2d$ and using (5.16), Lemma 5.6, (5.20), (5.21) and (5.24). \square

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